

Black Holes as Effective Geometries

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Abstract

Gravitational entropy arises in string theory via coarse graining over an underlying space of microstates. In this review we would like to address the question of how the classical black hole geometry itself arises as an effective or approximate description of a pure state, in a closed string theory, which semiclassical observers are unable to distinguish from the “naive” geometry. In cases with enough supersymmetry it has been possible to explicitly construct these microstates in spacetime, and understand how coarse-graining of non-singular, horizon-free objects can lead to an effective description as an extremal black hole. We discuss how these results arise for examples in Type II string theory on $\text{AdS}_5 \times S^5$ and on $\text{AdS}_3 \times S^3 \times T^4$ that preserve 16 and 8 supercharges respectively. For such a picture of black holes as effective geometries to extend to cases with finite horizon area the scale of quantum effects in gravity would have to extend well beyond the vicinity of the singularities in the effective theory. By studying examples in M-theory on $\text{AdS}_3 \times S^2 \times \text{CY}$ that preserve 4 supersymmetries we show how this can happen.

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1 Introduction

A spacetime geometry can carry an entropy in string theory via coarse graining over an underlying set of microstates. Since the initial success of string theory in accounting for the entropy of supersymmetric black holes by counting states in a field theory [2] there has been an ongoing effort to understand exactly what the structure of these microstates is and how they manifest themselves in gravity. It has been shown that in examples with enough supersymmetry, including some extremal black holes, one can construct a basis of “coherent” microstates whose spacetime descriptions in the $\hbar \rightarrow 0$ limit approach non-singular, horizon-free geometries which resemble a topologically complicated “foam”. Conversely, in these cases the quantum Hilbert space of states can be constructed by directly quantizing a moduli space of smooth classical solutions. Nevertheless, the *typical* states in these Hilbert spaces respond to semiclassical probes as if the underlying geometry was singular, or an extremal black hole. In this sense, these black holes are effective, coarse-grained descriptions of underlying non-singular, horizon-free states. We discuss how these results arise for states in Type II string theory on $\text{AdS}_5 \times S^5$ and on $\text{AdS}_3 \times S^3 \times T^4$ that preserve 16 and 8 supercharges respectively. We also discuss the connection between ensembles of microstates and coarse-grained effective geometries. Such results suggest the idea, first put forward by Mathur and collaborators [3,4], that *all* black hole geometries in string theory, even those with finite horizon area, can be seen as the effective coarse-grained descriptions of complex underlying horizon-free states¹ which have an extended

¹The idea here is that a single microstate does not have an entropy, even if its coarse-grained description in gravity has a horizon. Thus the spacetime realization of the microstate, having no entropy, should be in some sense horizon-free, even though the idea of a horizon, or even a geometry, may be difficult to define precisely at

spacetime structure. This idea seems initially unlikely because one might expect that the quantum effects that correct the classical black hole spacetime would be largely confined to regions of high curvature near the singularity, and would thus not modify the horizon structure. To study this we examine states of M-theory on $\text{AdS}_3 \times \text{S}^2 \times \text{CY}$ with 4 supercharges, where a finite horizon area can arise. We work with a large class of these states whose spacetime descriptions are amenable to study using split attractor flows and some of which give rise to “long throats” of the kind needed to give effective black hole behavior. These states are related to distributions of D-branes in spacetime. Surprisingly, it turns out that the quantized solution space has large fluctuations even at macroscopic proper distances, suggesting that the scale of quantum effects in gravity could extend beyond the vicinity of singularities in the effective theory. Thus, the idea that all black holes might simply be effective descriptions of underlying horizon-free objects tentatively survives this test.

1.1 Background

In string theory, black holes can often be constructed by wrapping D -branes on cycles in a compact manifold X so they appear as point like objects in the spatial part of the non-compact spacetime, $\mathbb{R}^{1,d-1}$. As the string coupling is increased, these objects backreact on spacetime and can form supersymmetric spacetimes with macroscopic horizons. The entropy associated with these objects can be determined “microscopically” by counting BPS states in a field theory living on the branes and this has been shown in many cases to match the count expected from the horizon area (see [2, 5] for the prototypical calculations). Although the field theory description is only valid for very small values of the string coupling g_s the fact that the entropy counting in the two regimes coincides can be attributed to the protected nature of BPS states that persist in the spectrum at any value of the coupling unless a phase transition occurs or a wall of stability is crossed². The fact that the (leading) contribution to the entropy of the black hole could be reproduced from counting states in a sector of the field theory suggests that the black hole microstates dominate the entropy in this sector.

While it is very helpful that these states can be counted at weak coupling, understanding the nature of these states in gravity at finite coupling remains an open problem. As g_s is increased the branes couple to gravity and we expect them to start backreacting on the geometry. The main tools we have to understand the spacetime or closed-string picture of the system are the AdS/CFT correspondence and the physics of D-branes.

Within the framework of the AdS/CFT correspondence black holes with near horizon geometries of the form $\text{AdS}_m \times \mathcal{M}$ must correspond to objects in a dual conformal field theory that have an associated entropy³. A natural candidate is a thermal ensemble or density matrix, in

a microscopic level.

²It is also possible that BPS states pair up and get lifted from the spectrum. For the systems considered in these notes, however, this does not appear to be an important phenomena.

³More generally objects in AdS with horizons, microscopic or macroscopic, are expected to have an associated

the CFT, composed of individual pure states (see e.g. [6]). AdS/CFT then suggests that there must be corresponding pure states in the closed string picture and that these would comprise the microstates of the black hole. It is not clear, however, that such states are accessible in the supergravity description. First, the dual objects should be closed string *states* and may not admit a classical description. Even if they do admit a classical description they may involve regions of high curvature and hence be inherently stringy. For BPS black holes⁴, however, we may restrict to the BPS sector in the Hilbert space where the protected nature of the states suggests that they might persist as we tune continuous parameters (barring phase transitions or wall crossings). We may then hope to see a supergravity manifestation of these states, and indeed this turns out to be the case for systems with sufficient supersymmetry. However, the large N limit⁵, which must be taken for supergravity to be a valid description, bears many similarities with the $\hbar \rightarrow 0$ limit in quantum mechanics where we know that most states do not have a proper classical limit. As we will see, if a supergravity description can be obtained at all, it will only be for appropriately “semiclassical” or “coherent” states.

Despite these potential problems, recently, a very fruitful program has been undertaken to explore and classify the smooth supergravity duals of coherent CFT states in the black hole ensemble. Smoothness here is important because if these geometries exhibit singularities we expect these to either be resolved by string-scale effects, making them inaccessible in supergravity, or enclosed by a horizon implying that the geometry corresponds, not to a pure state, but rather an ensemble with some associated entropy.

Large classes of such smooth supergravity solutions, asymptotically indistinguishable from black hole⁶ solutions, have indeed been found [7, 8, 9, 10, 11, 12, 13, 14] (and related [15, 16] to previously known black hole composites [17, 18, 19]). These are complete families of solutions preserving a certain amount of supersymmetry with fixed asymptotic charges⁷ and with no (or very mild) singularities.

In constructing such solutions it has often been possible to start with a suitable probe brane solution with the correct asymptotic charges in a flat background and to generate a supergravity

entropy which should manifest itself in the dual CFT.

⁴Here “BPS” can mean either 1/2, 1/4 or 1/8 BPS states or black holes in the full string theory. The degree to which states are protected depends on the amount of supersymmetry that they preserve and our general remarks should always be taken with this caveat.

⁵ N measures the size of the system. For black holes it is usually related to mass in the bulk and conformal weight in the CFT.

⁶Throughout this paper we will be discussing “microstates” of various objects in string theory but the objects will not necessarily be holes (i.e. spherical horizon topology) nor will they always have a macroscopic horizon. In fact, there is no 1/2 BPS solution in $\text{AdS}_5 \times \text{S}^5$ with any kind of a horizon. We will, none-the-less, somewhat carelessly continue to refer to these as “microstates” of a black hole for the sake of brevity.

⁷The question of which asymptotic charges of the microstates should match those of the black hole is somewhat subtle and depends on which ensemble the black hole is in. In principle some of the asymptotic charges might be traded for their conjugate potentials. Moreover, the solutions will, in general, only have the same isometries asymptotically.

solution by backreacting the probe [7, 10, 20]. In a near-horizon limit these back-reacted probe solutions are asymptotically AdS, and by identifying the operator corresponding to the probe and the state it makes in the dual CFT, the backreacted solution can often be understood as the spacetime realization of a coherent state in the CFT. Lin, Lunin and Maldacena [7] showed that the back-reaction of such branes (as well their transition to flux) was identified with a complete set of asymptotically AdS₅ supergravity solutions (as described above) suggesting that the latter should be related to 1/2 BPS states of the original $D3$ probes generating the geometry. Indeed, in [21, 22] it was shown that quantizing the space of such supergravity solutions as a classical phase space reproduces the spectrum of BPS operators in the dual $\mathcal{N} = 4$ superconformal Yang-Mills (at $N \rightarrow \infty$).

In a different setting Lunin and Mathur [9] were able to construct supergravity solutions related to configurations of a $D1$ - $D5$ brane in six dimensions (i.e. compactified on a T^4) by utilizing dualities that relate this system to an $F1$ - P system (see also [8]). The latter system is nothing more than a BPS excitation of a fundamental string quantized in a flat background. The back reaction of this system can be parametrized by a profile $F^i(z)$ in \mathbb{R}^4 (the transverse directions). T-duality relates configurations of this system to that of the $D1$ - $D5$ system.

Recall that the naive back-reaction of a bound state of $D1$ - $D5$ branes is a singular or “small” black hole in five dimensions. The geometries arising from the $F1$ - P system, on the other hand, are smooth after dualizing back to the $D1$ - $D5$ frame, though they have the same asymptotics as the naive solution [8]. Each $F1$ - P curve thus defines a unique supergravity solution with the same asymptotics as the naive $D1$ - $D5$ black hole but with different subleading structure. Smoothness of these geometries led Lunin and Mathur to propose that these solutions should be mapped to individual states of the $D1$ - $D5$ CFT. The logic of this idea was that *individual* microstates do not carry any entropy, and hence should be represented in spacetime by configurations without horizons. Lunin and Mathur also conjectured that the naive black hole geometry is somehow a coarse graining over all these smooth solutions, i.e. that the black hole itself is simply an effective, coarse-grained description. This idea sometimes goes under the name of *the fuzz ball proposal*.

The focus of these notes will be [23, 24, 25, 26, 27, 28, 29, 30, 31, 32] which use well-controlled supersymmetric examples to explore the idea that black holes might be simply “effective geometries”, i.e. that they are effective coarse-grained descriptions of underlying horizon-free objects. This will also involve understanding the nature of typical black hole microstates and how they may be resolved by probes [23]. The discussion will involve 1/2 BPS states in AdS₅ × S⁵, 1/4 BPS states of the $D1$ - $D5$ system, and the least controlled 1/8 BPS case where we will study bound multicenter configurations in four and five dimensions. Only in the final case will genuine macroscopic horizons be possible but the 1/2 and 1/4 BPS cases are under more technical control and hence important to study. In all cases we will try to understand how the BPS spectrum emerges in supergravity, how it is related to the BPS spectrum of a dual CFT (or more generally the brane theory) and how such states might contribute to the ensembles char-

acterizing black holes in string theory. Much of what will be discussed is a review of work by other authors, including [7, 3, 21, 10, 19, 12, 14].

As the related literature is voluminous and complicated we attempt to provide, in section 7, a brief survey of the various works and which branches of the field they fit into. This survey is by no means exhaustive and no doubt neglects many important works but we feel it may, nonetheless, serve as a useful reference for readers attempting to orient themselves within the field.

1.2 Some answers to potential objections

The idea that black holes are simply effective descriptions of underlying horizon-free objects is confusing because it runs counter to well-established intuitions in effective field theory, most importantly the idea that near the horizon of a large black hole the curvatures are small and hence so are the effects of quantum gravity. Indeed, it is not easy to formulate a precisely stated conjecture for black holes with finite horizon area, although for extremal black holes with enough supersymmetry a substantial amount of evidence has accumulated for the correctness of the picture, as reviewed in this article. To clarify some potential misconceptions, we transcribe below a dialogue between the authors, addressing some typical objections and representing our current point of view. Also see [4, 33, 34].

1. How can a smooth geometry possibly correspond to a “microstate” of a black hole?

Smooth geometries do *not* exactly correspond to states. Rather, as classical solutions they define points in the phase space of a theory (since a coordinate and a momenta define a history and hence a solution; see section 2.1 for more details) which is isomorphic to the solution space. In combination with a symplectic form, the phase space defines the Hilbert space of the theory upon quantization. While it is not clear that direct phase space quantization is the correct way to quantize gravity in its entirety this procedure, when applied to the BPS sector of the theory, seems to yield meaningful results that are consistent with AdS/CFT.

As always in quantum mechanics, it is not possible to write down a state that corresponds to a point in phase space. The best we can do is to construct a state which is localized in one unit of phase space volume near a point. We will refer to such states as coherent states. Very often (but not always, as we will see later in these notes) the limit in which supergravity becomes a good approximation corresponds exactly to the classical limit of this quantum mechanical system, and in this limit coherent states localize at a point in phase space. It is in this sense, and only in this sense, that smooth geometries can correspond to microstates. Clearly, coherent states are very special states, and a generic state will *not* admit a description in terms of a smooth geometry.

2. How can a finite dimensional solution space provide an exponential number of states?

The number of states obtained by quantizing a given phase space is roughly given by the volume of the phase space as measured by the symplectic form ω , $N \sim \int \omega^k / k!$ for a $2k$ -dimensional phase space. Thus, all we need is an exponentially growing volume which is relatively easy to achieve.

3. Why do we expect to be able to account for the entropy of the black hole simply by studying smooth supergravity solutions?

Well, actually, we do not really expect this to be true. In cases with enough supersymmetry, one does recover all BPS states of the field theory by quantizing the space of smooth solutions, but there is no guarantee that the same will remain true for large black holes, and the available evidence does not support this point of view. We do however expect that by including stringy degrees of freedom we should be able to accomplish this, in view of open/closed string duality.

4. If black hole “microstates” are stringy in nature then what is the content of the “fuzzball proposal”?

The content of the fuzzball proposal is that the closed string description of a generic microstate of a black hole, while possibly highly stringy and quantum in nature, has interesting structure that extends all the way to the horizon of the naive black hole solution, and is well approximated by the black hole geometry outside the horizon.

More precisely the naive black hole solution is argued to correspond to a thermodynamic ensemble of pure states. The *generic* constituent state will not have a good geometrical description in classical supergravity; it may be plagued by regions with string-scale curvature and may suffer large quantum fluctuations. These, however, are not restricted to the region near the singularity but extend all the way to the horizon of the naive geometry. This is important as it might shed light on information loss via Hawking radiation from the horizon as near horizon processes would now encode information about this state that, in principle, distinguish it from the ensemble average.

5. Why would we expect string-scale curvature or large quantum fluctuations away from the singularity of the naive black hole solution? Why would the classical equations of motion break down in this regime?

As mentioned in the answer to question 1, it is not always true that a solution to the classical equations of motion is well described by a coherent state, even in the supergravity limit. In particular there may be some regions of phase space where the density of

states is too low to localize a coherent state at a particular point. Such a point, which can be mapped to a particular solution of the equations of motion, is not a good classical solution because the variance of any quantum state whose expectation values match the solution will necessarily be large.

Another way to understand this is to recall that the symplectic form effectively discretizes the phase space into \hbar -sized cells. In general all the points in a given cell correspond to classical solutions that are essentially indistinguishable from each other at large scales. It is possible, however, for a cell to contain solutions to the equation of motion that *do* differ from each other at very large scales. Since a quantum state can be localized at most to one such cell it is not possible to localize any state to a particular point within the cell. Only in the strict $\hbar \rightarrow 0$ limit will the cell size shrink to a single point suggesting there might be states corresponding to a given solution but this is just an artifact of the limit. A specific explicit example of such a scenario is discussed later in these lecture notes.

Thus, even though the black hole solution satisfies the classical equation of motion all the way to the singularity this does not necessarily imply that when quantum effects are taken into account that this solution will correspond to a good semi-classical state with very small quantum fluctuations.

6. So is a black hole a pure state or a thermal ensemble?

In a fundamental theory we expect to be able to describe a quantum system in terms of pure states. This applies to a black hole as well. At first glance, since the black hole carries an entropy, it should be associated to a thermal ensemble of microstates. But, as we know from statistical physics, the thermal ensemble can be regarded as a technique for approximating the physics of the generic microstate in the microcanonical ensemble with the same macroscopic charges. Thus, one should be able to speak of the black hole as a coarse grained effective description of a generic underlying microstate. Recall that a typical or generic state in an ensemble is very hard to distinguish from the ensemble average without doing impossibly precise microscopic measurements. The entropy of the black hole is then, as usual in thermodynamics, a measure of the ignorance of macroscopic observers about the nature of the microstate underlying the black hole.

7. What does an observer falling into a black hole see?

This is a difficult question which cannot be answered at present. The above picture of a black hole does suggest that the observer will gradually thermalize once the horizon has been passed, but the rate of thermalization remains to be computed. It would be interesting to do this and to compare it to recent suggestions that black holes are the most efficient scramblers in nature [35, 36, 37].

8. Does the fuzzball proposal follow from AdS/CFT?

As we have defined it the fuzzball proposal does not follow from AdS/CFT. The latter is obviously useful for many purposes. For example, given a state or density matrix, we can try to find a bulk description by first computing all one-point functions in the state, and by subsequently integrating the equations subject to the boundary conditions imposed by the one-point function. If this bulk solution is unique and has a low variance (so that it represents a good saddle-point of the bulk path integral) then it is the right geometric dual description. In particular, this allows us to attempt to find geometries dual to superpositions of smooth geometries. What it does not do is provide a useful criterion for which states have good geometric dual descriptions; it is not clear that there is a basis of coherent states that all have decent dual geometric descriptions, and it is difficult to determine the way in which bulk descriptions of generic states differ from each other. In particular, it is difficult to show that generic microstates have non-trivial structure all the way up to the location of the horizon of the corresponding black hole.

9. To what degree does it make sense to consider quantizing a (sub)space of supergravity solutions?

In some instances a subspace of the solution space corresponds to a well defined symplectic manifold and is hence a phase space in its own right. Quantizing such a space defines a Hilbert space which sits in the larger Hilbert space of the full theory. Under some favourable circumstances the resulting Hilbert space may be physically relevant because a subspace of the total Hilbert space can be mapped to this smaller Hilbert space. That is, there is a one-to-one map between states in the Hilbert space generated by quantizing a submanifold of the phase space and states in the full Hilbert space whose support is localized on this submanifold.

For instance, in determining BPS states we can imagine imposing BPS constraints on the Hilbert space of the full theory, generated by quantizing the full solution space, and expect that the resulting states will be supported primarily on the locus of points that corresponds to the BPS phase space; that is, the subset of the solution space corresponding to classical BPS solutions. It is therefore possible to first restrict the phase space to this subspace and then quantize it in order to determine the BPS sector of the Hilbert space.

2 States, Geometries, and Quantization

Throughout these notes we will be exploring the relationship between families of smooth supergravity solutions and coherent microstates with the charges of supersymmetric black holes in string theory. The logic of this relationship is illustrated in Fig. 1.

The first important component is a (complete) family of supergravity solutions preserving the same supersymmetry and with the same asymptotic conserved charges (see footnote 7) as a BPS black hole. As solutions they can be related to points in a phase space and, as a family, they define a submanifold of the full phase space (this notion is elaborated upon in the next section and references therein). Because they are BPS they can be argued to generate a proper decoupled phase space of their own⁸ [21, 22], at least for the purpose of enumerating states. Indeed, in the cases we consider, one can check that the restriction of the symplectic form to this space is non-degenerate implying that the space is actually a symplectic submanifold⁹. For a thorough discussion of the subtleties involved in this “on-shell quantization” the reader is referred to [22, Sec. 2.7-2.10]. Quantizing the space of such solutions as a phase space yields a Hilbert space populated by putative BPS microstates of the black hole. In [21] this was done for 1/2 BPS geometries asymptotic to $\text{AdS}_5 \times S^5$ and found to reproduce the 1/2 BPS spectrum of the dual CFT. In general, though, we would expect that supergravity alone would not suffice and that stringy degrees of freedom would be necessary to reproduce the complete spectrum of BPS states.

The states arising from quantizing only smooth supergravity solutions, without any stringy degrees of freedom, should thus correspond to some restricted subspace of the full BPS Hilbert space of the theory (which corresponds to the BPS sector of the CFT). We would like to understand how these pure states relate to black holes. The latter have entropy so we expect them to be dual to density matrices in the CFT. The ensemble of states making up such a density matrix should be a suitable thermodynamic average over pure states in the same sector of the Hilbert space as the black hole. As our “microstates” arise from geometries that preserve the same supersymmetries and carry the same asymptotic charges as the black hole, they provide suitable candidates for states in the ensemble. However, it is likely that the generic state in this ensemble will not be semi-classical and thus will only be accessible in the CFT or by directly quantizing the BPS phase space. It is also not clear whether a large portion of the states in the black hole ensemble, or indeed any typical states at all, arise from quantizing only supergravity modes. As mentioned above this has proven to be the case when sufficient supersymmetry is present but it is not clear if it will continue to do so for less supersymmetric black holes. Nevertheless, as the states arising from supergravity are relatively accessible (compared to more stringy states) it is important to explore their structure and also to determine how they are related to the typical states making up the black hole ensemble.

In some cases, there is reason to believe that the Hilbert space enjoys a more refined decomposition in subsectors than just by macroscopic quantum numbers alone. In [38, 39, 40], for instance, a decomposition based on split attractor trees is conjectured (this will be discussed

⁸Though these arguments are for the 1/2 and 1/4 BPS case they should extend to 1/8 BPS bearing in mind possible discontinuities in the spectrum at walls of marginal stability.

⁹As noted in [22] this is related to the fact that the solutions are stationary but not static so the momenta conjugate to the spatial components of the metric are non-vanishing.

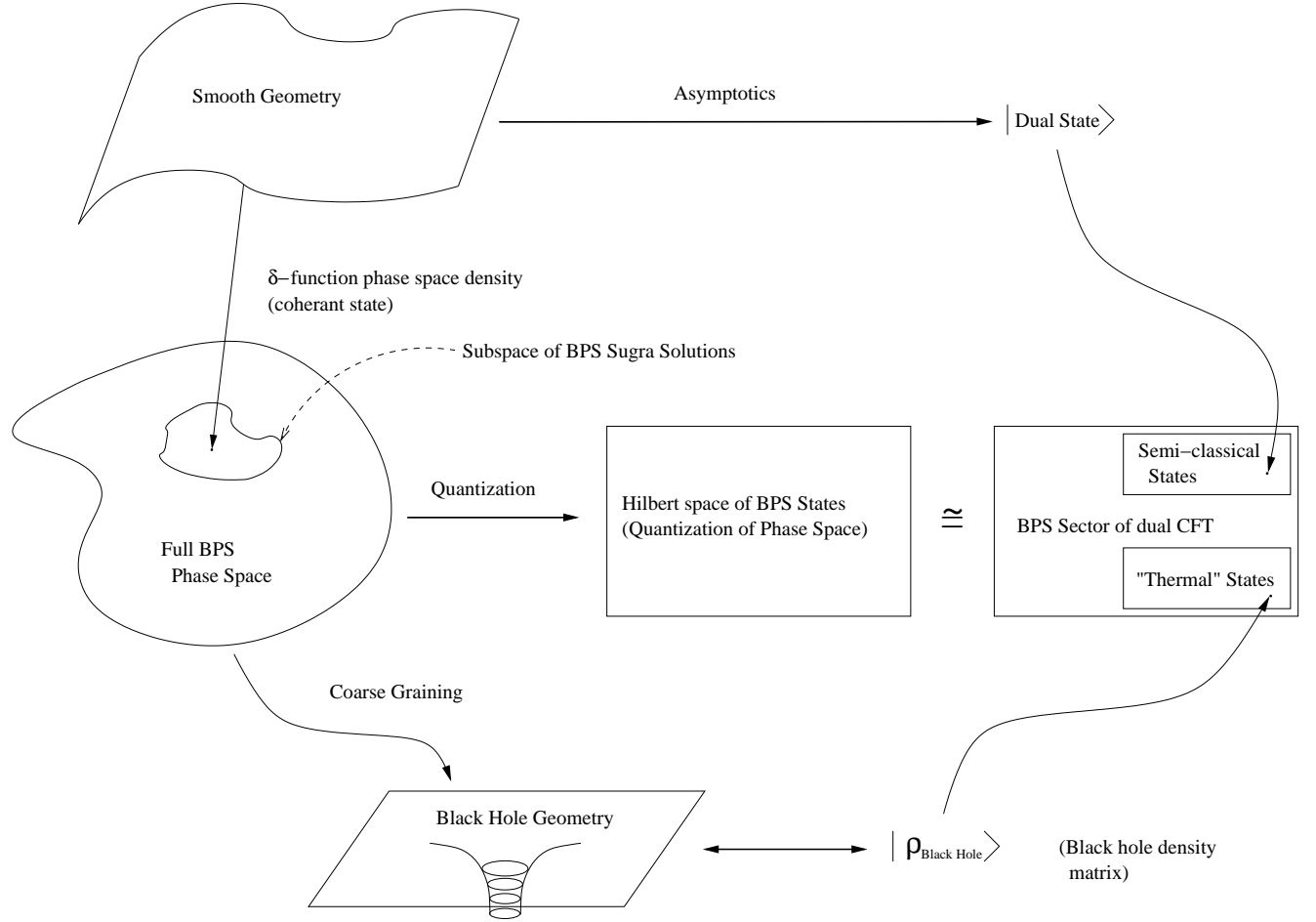


Figure 1: Relationship between various components appearing in the study of black hole microstates and black holes as effective geometries. The smooth geometries making up the phase space can be thought of either as classical solutions defining a solution space (isomorphic to a phase space) or as highly-localized phase space densities corresponding to coherent states. The black hole at the bottom of the figure is then to be generated by coarse graining (in some suitable sense) over a large number of underlying horizon-free configurations; the resultant geometry need not be a black hole but may, for instance, include a naked singularity. The details can differ significantly between examples – the quantization, for instance, is rather different for the 1/4 and 1/8 BPS case. The “thermal states” in the BPS sector box refer to ensembles with any chemical potential that couples to an operator that commutes with supersymmetry, and thus acts within the BPS Hilbert space (for example the left-moving temperature in a 2d CFT while the right-movers are kept in their ground state).

further in section 5.3). In these cases it is possible that of all the states with the correct asymptotic charges the black hole ensemble will only include microstates from a single subsector. In [26] another (related) constraint on the constituents of the black hole ensemble was found. There it was argued that any geometries that do not survive a near-horizon decoupling limit should not contribute states to the black hole ensemble because they do not correspond to bona fide bound configurations of the original D -brane system generating the black hole.

In order to study the spacetime structure of the microstates of a black hole it is desirable to have an inverse map between the states in the CFT and classical geometries. Of course this map is not injective as many states in the CFT are not semi-classical so we would also like a criteria for determining which such states yield good classical geometries and which yield geometries with large quantum fluctuations. Possible criteria were discussed in [28,41] and will play a role in some of the arguments that follow. The point of view that we would like to assume is based on the need for a classical observer to measure the system [23]. Thus we would like to identify a set of operators, \mathcal{O}_α , in the CFT that are dual to “macroscopic observables”. The requirement that a state yield a good classical geometry can be translated into a constraint on the variance of the expectation values for these observables in the semi-classical limit.

2.1 Phase Space Quantization

The space of classical solutions of a theory is generally isomorphic to its classical phase space¹⁰. Heuristically, this is because a given point in the phase space, comprised of a configuration and associated momenta, can be translated into an entire history by integrating the equations of motion against this initial data; likewise, by fixing a spatial foliation, any solution can be translated into a unique point in the phase space by extracting a configuration and momentum from the solution evaluated on a fixed spacial slice. This observation can be used to quantize the theory using a symplectic form, derived from the Lagrangian, on the space of solutions rather than on the phase space. This is an old idea [42] (see also [43] for an extensive list of references and [44] [45] [46] for more recent work) which was used in [21,22] [47,48] to quantize the LLM [7] and Lunin-Mathur [9] geometries. An important subtlety in these examples is that it is not the entire solution space which is being quantized but rather a subspace of the solutions with a certain amount of supersymmetry.

In general, quantizing a subspace of the phase space will not yield the correct physics as it is not clear that the resultant states do not couple to states coming from other sectors. It is not even clear that a given subspace will be a symplectic manifold with a non-degenerate symplectic pairing. As discussed in [22] we expect the latter to be the case only if the subspace contains

¹⁰It is not entirely clear, however, which solutions should be included in defining the phase space. For instance, in treating gravity, it is not clear if trajectories which eventually develop singularities should also be included as points in the phase space or only solutions which are eternally smooth. As we will primarily be concerned with static or stationary solutions in these notes we will largely avoid this issue.

dynamics; for gravitational solutions we thus expect stationary solutions, for which the canonical momenta are not trivial, to possibly yield a non-degenerate phase space. This still does not address the issue of consistency as states in the Hilbert space derived by quantizing fluctuations along a constrained submanifold of the phase space might mix with modes transverse to the submanifold. When the submanifold corresponds to the space of BPS solutions one can argue, however, that this should not matter. The number of BPS states is invariant under continuous deformations that do not cross a wall of marginal stability or induce a phase transition. Thus if we can quantize the solutions in a regime where the interaction with transverse fluctuations is very weak then the energy eigenstates will be given by perturbations around the states on the BPS phase space, and, although these will change character as parameters are varied the resultant space should be isomorphic to the Hilbert space obtained by quantizing the BPS sector alone. If a wall of marginal stability is crossed states will disappear from the spectrum but there are tools that allow us to analyze this as it occurs (see section 5.3).

Let us emphasize that the validity of this decoupling argument depends on what questions one is asking. If we were interested in studying dynamics then we would have to worry about how modes on the BPS phase space interact with transverse modes. For the purpose of enumerating or determining general properties of states, however, as we have argued, it should be safe to ignore these modes. For an example of the relation between states obtained by considering the BPS sector of the fully quantized Hilbert space and the states obtained by quantizing just the BPS sector the phase space see [18] and [27].

As mentioned, the LLM and Lunin-Mathur geometries have already been quantized and the resultant states were matched with states in the dual CFTs. We will have occasion to mention this briefly in the sequel but we will ultimately focus on the quantization of $\mathcal{N} = 2$ solutions in four (or $\mathcal{N} = 1$ in five) dimensions. For such solutions, although a decoupling limit has been defined [27], the dual $\mathcal{N} = (0, 4)$ CFT is rather poorly understood. Thus quantization of the supergravity solutions may yield important insights into the structure of the CFT and will be important in studying the microstates of the corresponding extremal black objects.

2.2 Black holes, AdS throats and dual CFTs

One of the most powerful tools to study properties of black holes in string theory is the AdS/CFT correspondence [49]. This conjecture relates string theory on backgrounds of the form $\text{AdS}_{p+1} \times \mathcal{M}$ to a CFT _{p} that lives on the boundary of the AdS_{p+1} space. Such backgrounds arise from taking a particular decoupling limit of geometries describing black objects such as black holes, black strings, black tubes, etc. This limit amounts to decoupling the physics in the near horizon region¹¹ of the black object from that of the asymptotically flat region by

¹¹In some of the cases treated in these notes the region will not be an actual near-horizon region as the original solutions may be horizon-free (or, in some cases, may have multiple horizons) but the decoupling limits are motivated by analogy with genuine black holes where the relevant region is the near horizon one.

scaling the appropriate Planck length, l_p , to decouple the asymptotic gravitons from the bulk. At the same time one needs to scale appropriate spatial coordinates with powers of l_p to keep the energies of some excitations finite. This procedure should be equivalent to the field theory limit of the brane bound states generating the geometry under consideration.

We are interested in black objects which describe normalizable deformations in the AdS_{p+1} background. These correspond to a state/density matrix on the dual CFT according to the following dictionary

BULK	BOUNDARY
$\exp(-S_{\text{bulk}}^{\text{on shell}})$	$\text{Tr}[\rho \mathcal{O}_1 \dots \mathcal{O}_n] = \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle_\rho$
classical geometries	semiclassical states ?
black hole	$\rho \sim \exp\{-\sum_i \beta_i \mathcal{O}_i\}$
entropy S	$S = -\text{Tr}(\rho \log \rho)$
bulk isometry D	$[\rho, \hat{D}] = 0$
ADM quantum numbers of D	$\text{Tr}(\rho \hat{D}) = \langle \hat{D} \rangle = D_{\text{ADM}}$

In the first line \mathcal{O}_i are operators dual to sources turned on in the boundary. They are included in the bulk calculation of $(-S_{\text{bulk}}^{\text{on shell}})$. The second line can be seen as the definition of the dual semiclassical state. More specifically, a semiclassical state is the one that has an unambiguous dual bulk geometry (i.e. that in the classical limit ($N \rightarrow \infty$ and $\hbar \rightarrow 0$) macroscopic observables take on a fixed expectation value with vanishing variance). In some ideal situations such semiclassical states turn out to be the analog of coherent states in the harmonic oscillator. In the third line, we describe a typical form of a density matrix that we expect to describe black holes. This form is motivated by the first law of thermodynamics: the entropy as defined in the fourth line obeys $dS = \sum_i \beta_i d\langle \mathcal{O}_i \rangle$, and by matching this to the first law as derived from the bulk description of the black hole we can identify the relevant set of operators \mathcal{O}_i and potentials μ_i and guess the corresponding density matrix. The fourth line simply states that we expect a relation between the bulk and boundary entropies. In the fifth and the last line, \hat{D} is the current/operator dual to the bulk isometry D.

One question that we want to shed some light on in these notes is; “*Given a density matrix ρ on the CFT side, is there a dual geometry in the bulk?*”. On general grounds one could have expected that a general density matrix ρ should be dual to a suitably weighted sum over geometries, each of which could be singular, have regions with high curvature, and perhaps not have good classical limits. As a result the dual gravitational description of a general density matrix will not generally be trustworthy. However, under suitable circumstances, it can happen that there is a dual “*effective*” geometry that describes the density matrix ρ very well. This procedure of finding the effective geometry is what we will call “*coarse graining*”. In the gravity description, this amounts to neglecting the details that a classical observer cannot access anyway due to limitations associated to the resolution of their apparatus. So, one can

phrase our question in the opposite direction, “*What are the characteristics of a density matrix on the CFT side, so that there is a good dual effective geometry that describes the physics accurately?*”.

One can try to construct the dual effective geometry following the usual AdS/CFT prescription. To do so, one should first calculate all the non vanishing expectation values of all operators dual to supergravity modes (assuming one knows the detailed map between the two). On the CFT side, these VEVs are simply given by

$$\langle \mathcal{O}_i \rangle = \text{Tr}(\rho \mathcal{O}_i),$$

and they determine the boundary conditions for all the supergravity fields. The next step is to integrate the gravity equations of motion subject to these boundary conditions to get the dual geometry. This is in principle what has to be done according to AdS/CFT prescription. The problem with this straightforward approach is that it is not terribly practical, and we will therefore revert to a different approach¹². Before describing various examples in more detail, we first describe the main idea in general terms. We will first start by describing the connection between quantum physics and the classical phase space. After that, we are going to briefly describe the philosophy behind constructing effective geometries.

2.3 Phase Space Distributions

To have an idea about what it means to average over ensembles of geometries, or “coarse grain” as we will refer to it, we need to understand some general features of the bulk theory. In general, we will assume that we are dealing with a supergravity theory in the bulk¹³. Recall that solutions to the supergravity equation of motion can be associated with points in a phase space (see for example [46]). The boundary theory, on the other hand, is generally studied as a quantum conformal field theory. As a result we are looking for a map between quantum states (CFT) and classical objects living in a phase space (bulk). A well known example of such a map is the map between quantum states and their corresponding classical phase space densities (see the review [50] and references there in to the original literature). A good guess then is that the map that we are looking for should involve a “dressed” version of the phase space densities of quantum states in the CFT. Let us pause for a moment to discuss the idea of a phase space distribution [50]. A particle (or statistical system) in a quantum theory is described by giving its density matrix ρ . The result of any measurement can be seen as an expectation value of an appropriate operator which is given explicitly by

$$\langle \mathcal{O} \rangle_\rho = \text{Tr}(\rho \mathcal{O}) \tag{2.1}$$

¹²Though it would be interesting to study in some detail the connection between the two.

¹³Although we would like to extend this discussion to include stringy degrees of freedom (when relevant) we do not, at present, have any control over the latter.

This is reminiscent of classical statistical mechanics where the measurements are averages of appropriate quantities using some statistical distribution

$$\langle \mathcal{O} \rangle_w = \int dp dq w(p, q) \mathcal{O}(p, q) \quad (2.2)$$

where the integration is over the full phase space. One can wonder at this point if it is possible to construct a density $w(p, q)$ so that one can rewrite equation (2.1) as equation (2.2). The answer is affirmative: for every density matrix ρ there is an associated phase space distribution w_ρ such that for all operators A the following equality holds

$$\int dp dq w_\rho(p, q) A(p, q) = \text{Tr} (\rho A(\hat{p}, \hat{q})) \quad (2.3)$$

What about the uniqueness of w_ρ ? Recall that in a quantum theory we have to face the question of operator ordering. This comes about because the operators \hat{q} and their dual momenta \hat{p} don't commute with each other. This means that the distribution w_ρ should somehow include information about the chosen order of \hat{p} and \hat{q} . As a result there does not exist a unique phase space distribution. For example, the distribution corresponding to Weyl ordering is the Wigner distribution, which is given by

$$w(p, q) \sim \int dy \langle q - y | \rho | q + y \rangle e^{2ipy} . \quad (2.4)$$

This distribution suffers from the fact that it is not positive definite in general. It is also quite sensitive to the physics at a quantum scale [50, 23] as it usually has fluctuations of order \hbar . Another drawback of this distribution is that it is difficult to work with from a computational standpoint. There is another commonly used distribution which is positive definite: the Husimi distribution. It is roughly the convolution of the Wigner distribution with a Gaussian. This eliminates most of the fluctuations of order \hbar . The price that one pays for this is that the resulting operators must be anti-normal ordered. However, for semi-classical states, which by definition are states for which the classical limit is unambiguous, $w_\rho(p, q)$ should be independent of the choice of ordering prescription in the classical limit as well, so this is not actually much of a problem.

2.4 Typical states versus coarse grained geometry

Let us recapitulate what we have discussed so far and what the missing steps are to achieve our goal. On the gravity side we have geometries with certain asymptotics that in principle yield the one point functions of the dual operators in the CFT. On top of that we have in principle a way to quantize the reduced phase space of solutions by using the induced symplectic form. On the CFT side we can consider individual states or density matrices ρ and find the corresponding expectation values of operators. The only missing ingredient is to construct the dual effective

geometry. To do so, we expect to use some sort of “dressed” version of the CFT phase space densities described above. But how do we carry out the coarse graining that is presumably needed to reach a description in a bulk classical geometry? At this stage we have two options: (i) we first select a typical representative from the CFT ensemble of states and then map this typical state directly to geometry, or (ii) we somehow average over all the geometries dual to the states in the ensemble.

2.4.1 Typical states/geometries

A typical state in an ensemble is one for which the expectation values of macroscopic observables agree to within the observable accuracy with the average of the observable in the entire ensemble. Obviously, this definition depends on the appropriate notions of macroscopic observables and observable accuracy, but in the examples we describe we will usually have a reasonably educated guess regarding what the typical states are. Given a typical state, we can try to map it directly to a solution of supergravity (this may still be a formidable task), after which one still needs to verify that the resulting geometry has no pathologies. This approach was followed for example in [4, 23].

2.4.2 Average/coarse graining

Alternatively, we can try to average over states and geometries directly. On the CFT side this is trivial since it essentially involves constructing a density matrix and following the usual rules of quantum mechanics. But the coarse graining procedure is difficult to implement on the bulk side because gravity is a non-linear theory. However, in all examples that we will study, the equations of motion of supergravity in the BPS sector will effectively be linearized, which allows us to solve the equations in terms of harmonic functions with sources. In addition, the space of solutions will be in one-to-one correspondence with distributions of the sources. This immediately suggests a suitable coarse-graining procedure: we simply smear the harmonic functions against the phase space density which describes the density matrix in question. This approach was followed for example in [25]. It would be interesting to explore in more detail whether this method gives rise to the appropriate averaging of the one-point functions, and to what extent it agrees with the approach based on typical states that we described in the previous paragraph.

2.5 Entropic suppression of variances

One might object any picture where the microstates of a black hole are individually realized in spacetime as extended horizon-free bound states would inevitably lead to radically different results for probe measurements. If this were so, then the usual black hole could not be a good effective description, and there would be a massive violation of our usual expectations from

effective field theory. Fortunately, one can show that in *any* scenario where the entropy of a black hole has a statistical interpretation in terms of states in a microscopic Hilbert space, the variance of finitely local observables over the Hilbert will be suppressed by a power of e^{-S} [31].

To see this, consider a quantum mechanical Hilbert space of states with energy eigenvalues lying between E and $E + \Delta E$ with a basis

$$\mathcal{M}_{bas} = \{ |s\rangle : H|s\rangle = e_s|s\rangle \quad ; \quad E \leq e_s \leq E + \Delta E \} . \quad (2.5)$$

Thus this sector of the Hilbert space consists of states

$$\mathcal{M}_{sup} = \left\{ |\psi\rangle = \sum_s c_s^\psi |s\rangle \right\} , \quad (2.6)$$

with $|s\rangle$ as in (2.5) and $\sum_s |c_s|^2 = 1$. The expectation value of the Hamiltonian H in any state in \mathcal{M}_{sup} also lies between E and $E + \Delta E$. If entropy of the system is $S(E)$, then the basis in (2.5) has dimension $e^{S(E)}$:

$$1 + \dim \mathcal{M}_{sup} = |\mathcal{M}_{bas}| = e^{S(E)} . \quad (2.7)$$

Now take \mathcal{O} to be any local operator and consider finitely local observables of the form

$$c_\psi = \langle \psi | \mathcal{O} | \psi \rangle \quad (2.8)$$

We would like to measure how these observables vary over the ensemble \mathcal{M}_{sup} . The ensemble averages of the observables (2.8) and their variances over the ensemble are given by

$$\langle c \rangle_{\mathcal{M}_{sup}} = \int D\psi \, c_\psi \quad (2.9)$$

$$\text{var}[c]_{\mathcal{M}_{sup}} = \int D\psi \, (c_\psi)^2 - \langle c \rangle_{\mathcal{M}_{sup}}^2 . \quad (2.10)$$

The differences between states in the ensemble of microstates in their responses to local probes are quantified by the standard-deviation to mean ratios

$$\frac{\sigma[c]_{\mathcal{M}_{sup}}}{\langle c \rangle_{\mathcal{M}_{sup}}} = \frac{\sqrt{\text{var}[c]_{\mathcal{M}_{sup}}}}{\langle c \rangle_{\mathcal{M}_{sup}}} . \quad (2.11)$$

It was shown in [31] that

$$\text{var}[c]_{\mathcal{M}_{sup}} < \frac{1}{e^S + 1} \text{var}[c]_{\mathcal{M}_{bas}} \quad (2.12)$$

where the variance on the right hand side is computed just over a set of basis elements while the variance on the left hand side is over the entire Hilbert space. This result follows because the generic state in the Hilbert space is a random superposition of the form (2.6), and in the computation of correlation functions the phases in the coefficients c_s^ψ lead to cancellations. Thus the only avenue to having a variance large enough to distinguish microstates by defeating

the e^S suppression in (2.12) is to find probe operators that have exponentially large correlation functions. Finitely local correlation functions in real time typically do not grow in this way and hence, for black holes, with their enormous entropy, a semiclassical observer will have no hope of telling microstates apart from each other. This is especially so because, as we will discuss in subsequent section, even the elements of the basis of microstates (2.5) for a black hole will usually possess the property of typicality, namely that they will be largely indistinguishable using coarse probes.

Thus, even if the microstates of a black hole are realized in spacetime as some sort of horizon free bound states, finitely local observables with finite precision, of the kind that are accessible to semiclassical observers, would fail to distinguish between these states. Indeed, the semiclassical observer, having finite precision, might as well take an ensemble average of the observables over the microstates, as this would give the same answer. The ensemble of microstates gives a density matrix with entropy S , and will be described in spacetime as a black hole geometry. In this sense, the black hole geometry will give the effective description of measurements made by semiclassical observers.

3 $\text{AdS}_5 \times \text{S}^5$

We start with the best understood case, $\text{AdS}_5 \times \text{S}^5$, whose AdS/CFT dictionary is well developed. The dual CFT is $\mathcal{N} = 4$ $\text{SU}(N)$ super Yang-Mills where N is the number of D3 branes that generate the geometry. Many supergravity solutions that asymptote to $\text{AdS}_5 \times \text{S}^5$ are known, including ones with 1/2, 1/4, 1/8 and 1/16 of the original supersymmetries preserved. Black holes with a macroscopic horizon only exist either in the 1/16 BPS case [51] or without any supersymmetry. The latter include AdS-Schwarzschild black holes and some of their qualitative properties can be reproduced from the dual CFT [23]. However, we are going to restrict ourselves to the 1/2-BPS case, where completely explicit descriptions of both supergravity solutions [52, 53, 54, 7] as well as the CFT states [55, 56, 57] are known. Therefore, we can test the general philosophy that we have been advocating by examining the relationship between very heavy 1/2-BPS states in the CFT (conformal dimension of $O(N^2)$) [55, 56, 57], smooth gravitational solutions with the same energy [7], and the 1/2-BPS extremal black hole in AdS_5 [54]. Our exposition will be brief; see [23, 32, 28] for details and further references.

3.1 The 1/2-BPS sector in field theory.

The Hilbert space of 1/2-BPS states in $\mathcal{N} = 4$ $\text{U}(N)$ super Yang-Mills is isomorphic to the Hilbert space of N fermions in a harmonic oscillator potential as shown in [56, 57]. A basis for this Hilbert space can be enumerated in terms of Young diagrams with N rows as follows. The ground state is composed of fermions (labelled by $i = 1, \dots, N$) with energies $E_i^g = [(i-1)+1/2]\hbar$;

this is the Fermi sea of the system.¹⁴ When we excite these fermions, the energies become $E_i = (e_i + 1/2)\hbar$ for some positive disjoint integers $e_i \geq i - 1$. Because we are dealing with fermion wave functions that are completely antisymmetrized we can always order $\{e_i\}$ in an ascending order $e_1 < e_2 < \dots < e_N$. As a result, the numbers r_i defined by

$$r_i = e_i - i + 1$$

form a non-decreasing set of integers which can be encoded in a Young diagram where r_i describes the length of the i -th row. It is convenient to also introduce variables c_j which count the number of columns of length j [58]. They are related to the r_i via

$$c_N = r_1, \quad c_{N-i} = r_{i+1} - r_i, \quad i = 1, 2, \dots, (N-1)$$

and clearly

$$r_{i+1} = e_{i+1} - i = c_{N-i} + \dots + c_N.$$

A property of Young diagrams that will be useful later is that a single Young diagram corresponds to a geometry with $U(1)$ symmetry in the bulk. This comes about because a single Young diagram is associated to a density matrix of a pure state, i.e. of the form

$$\rho = |\psi\rangle\langle\psi|,$$

where $|\psi\rangle$ has a fixed energy eigenvalue (simply given by the total number of boxes). The density matrix therefore commutes with the Hamiltonian which generates rotations in phase space, and according to the table in section 2.2 the corresponding supergravity solution should also possess this rotational invariance.

3.2 The typical very heavy state

Large classical objects in AdS_5 have masses of order N^2 , and hence 1/2-BPS states that correspond to black hole microstates will have conformal dimensions of $O(N^2)$. In the fermion language above, such states have a total excitation energy of $O(N^2)$, or, equivalently, $O(N^2)$ boxes in the corresponding Young diagram. We would like to study such very heavy states in the semiclassical limit ($\hbar \rightarrow 0$ with $\hbar N$ fixed) and understand their relation to the 1/2-BPS black hole [54] and the smooth 1/2-BPS geometries [7].

Highly excited states of the large N free fermion system can reliably be discussed in a canonical ensemble in which temperature rather than energy is held fixed¹⁵. Thus we write a

¹⁴We set the frequency of the harmonic oscillator $\omega = 1$.

¹⁵Often, it is also interesting to study more general canonical ensembles where additional “chemical potentials” are included such as the angular velocity coupling to angular momentum, or electrostatic potentials coupling to charges. We will see some examples in the case of $AdS_3 \times S^3$. Also note the temperature here is not physical – it is simply a Lagrange multiplier used to fix the energy of states included in this 1/2-BPS ensemble.

partition function

$$Z = \sum_{c_1, c_2, \dots, c_N=1}^{\infty} e^{-\tilde{\beta}\omega \sum_j j c_j} = \prod_{j=1}^N \frac{1}{1 - e^{-\beta j}} \equiv \prod_{j=1}^N \frac{1}{1 - q^j} \quad (3.1)$$

where we dropped the zero point energy $e^{\tilde{\beta}\hbar\omega N^2/2}$ and set $\beta = \tilde{\beta}\hbar$ and $q = e^{-\beta}$. By fixing

$$\beta \sim \frac{1}{\sqrt{\Delta}} \quad (3.2)$$

one studies an ensemble of operators with typical conformal dimension of $O(\Delta)$. We will take $\Delta \sim N^2$ since we are interested in states with energies in the range of the large extremal black hole [54]. In this range the entropy of the ensemble scales as

$$S \sim \sqrt{\Delta} \sim N \quad (3.3)$$

which grows in the large N limit, but not fast enough to give rise to a finite classical horizon area in the dual gravity picture.

Using the canonical ensemble one can easily calculate the average energy as well as the expectation value and standard deviation of c_i :

$$\langle c_j \rangle = \frac{q^j}{1 - q^j} \quad ; \quad \frac{\sigma(c_j)}{\langle c_j \rangle} = \left(\frac{1}{\sqrt{\langle c_j \rangle}} \right) \frac{1}{\sqrt{1 - q^j}}. \quad (3.4)$$

In the thermodynamic limit $N \gg 1$ where one rescales the rows and columns of the Young diagram by factor of \sqrt{N} , the Young diagram of a state in the ensemble approaches a certain *limiting shape* with probability 1 [59]. In other words, in the large N limit almost all states/operators belonging to the canonical ensemble under study will have associated Young diagrams that have vanishingly small fluctuations around this limit curve. One can check this claim by evaluating the standard deviation to mean ratio above in the large N limit [23]. The limit shape describes a “*typical*” Young diagram in this ensemble (see section 2.4).

To describe this limiting shape in some more detail, let us introduce two coordinates x and y along the rows and columns of the Young diagram. We adopt the convention where the origin $(0, 0)$ is the bottom left corner of the diagram, and x increases going up while y increases to the right. In fermion language, x labels the particle number and y its excitation above the vacuum. One then has

$$y(x) = \langle \mathbf{y}(x) \rangle = \sum_{i=N-x}^N \langle c_i \rangle \quad (3.5)$$

In the large N limit, x and y can be treated as continuous variables and the summation above becomes an integral. Since the c_i are independent random variables in the canonical ensemble, it is straightforward to evaluate $y(x)$ explicitly, and one obtains an equation for the limit shape of the form [23]

$$(1 - q^N)q^y + q^{N-x} = 1. \quad (3.6)$$

In sum, in the semiclassical limit ($\hbar \rightarrow 0$ with $\hbar N$ fixed), nearly all the elements of a basis of half-BPS states of $\mathcal{N} = 4$, $SU(N)$ Yang-Mills theory lie close to the typical state described by (3.6). As we will see, because of the structural similarity of all of these states to each other, a semiclassical observer will not be able to tell them apart.

3.3 The 1/2-BPS sector in supergravity

All 1/2-BPS solutions in $\text{AdS}_5 \times S^5$ supergravity are given by the LLM geometries [7]

$$\begin{aligned} ds^2 &= -h^{-2}(dt + V_i dx^i)^2 + h^2(d\eta^2 + dx^i dx^i) + \eta e^G d\Omega_3^2 + \eta e^{-G} d\tilde{\Omega}_3^2 \\ h^{-2} &= 2\eta \cosh G, \quad \eta \partial_\eta V_i = \epsilon_{ij} \partial_j z, \quad \eta(\partial_i V_j - \partial_j V_i) = \epsilon_{ij} \partial_\eta z \\ z &= \frac{1}{2} \tanh G, \quad z(\eta, x_1, x_2) = \frac{\eta^2}{\pi} \int dy_1 dy_2 \frac{\frac{1}{2} - u(0; y_1, y_2)}{[(\vec{x} - \vec{y})^2 - \eta^2]} \end{aligned} \quad (3.7)$$

where $i = 1, 2$. In these coordinates taken from [7], y_1 and y_2 have dimensions of length *squared*. In addition there is a self-dual 5-form field strength that depends on the function z . It is clear from the above equations that the full geometry is specified by choosing a boundary function $u(0; y_1, y_2)$. The requirement of smoothness of the geometry forces $u \in \{0, 1\}$. So one can see the function u as defining a droplet in the y_1, y_2 -plane whose boundary separates the region where $u = 1$ from the region where $u = 0$. This means smooth 1/2 BPS geometries are in one to one correspondence with droplets on the (y_1, y_2) -plane. 1/2-BPS geometries in which $0 < u < 1$ have null singularities and one of the goals is to understand the relation, if any, between such geometries and the smooth geometries and the CFT microstates discussed above.¹⁶ If u takes a constant value between 0 and 1 within a disk in the y_1, y_2 plane, it can be shown that the corresponding geometry is precisely the “superstar” of [52, 53, 54], a 1/2-BPS extremal black hole.

Following [7], in order to match these solutions with states in the field theory, consider geometries for which the regions in the (y_1, y_2) -plane where $u = 1$ are compact. Quantization of the flux in the geometry leads to the following identifications

$$\hbar \leftrightarrow 2\pi l_p^4, \quad N = \int \frac{d^2 y}{2\pi \hbar} u(0; y_1, y_2) \quad (3.8)$$

The conformal dimension¹⁷ Δ of a given configuration is

$$\Delta = \frac{1}{2} \int \frac{d^2 y}{2\pi \hbar} \frac{y_1^2 + y_2^2}{\hbar} u(0; y_1, y_2) - \frac{1}{2} \left(\int \frac{d^2 y}{2\pi \hbar} u(0; y_1, y_2) \right)^2 \quad (3.9)$$

The formulas above suggest that we should interpret $u(0; y_1, y_2)$ as a phase space density for a harmonic oscillator where y_1 and y_2 are treated as phase space coordinates. Indeed, this

¹⁶If $u > 1$ or $u < 0$ the geometry develops pathologies such as closed time-like curves, so we will exclude these [60].

¹⁷The bulk interpretation of the conformal dimension is energy.

function has a remarkably simple interpretation [7] in terms of the hydrodynamic limit of the phase space of the dual fermionic system, once we identify the y_1, y_2 -plane with the single particle phase space of the fermions. This has been confirmed by directly quantizing the phase space of smooth gravitational solutions [61, 21, 62], a procedure which explicitly recovers the “fermions in a harmonic oscillator” description of 1/2-BPS states that is derived from the dual field theory. This surprising result is obtained despite the fact that the configuration space of smooth solutions formally has structures at the string scale – presumably the large amount of supersymmetry is responsible for the success in reproducing the full quantum description from the restricted quantization of only smooth solutions in gravity. According to the general strategy, we should therefore try to identify $u(0; y_1, y_2)$ directly with the one-particle phase space density for any density matrix in the quantum-mechanical fermion system [23]. Specifically we identify

$$y_1, y_2 \leftrightarrow p, q \quad ; \quad u(0, y_1, y_2) \equiv 2\pi\hbar w(p, q) \quad (3.10)$$

where p, q are coordinates of the fermion one-particle phase space and $w(p, q)$ is a phase space density which encodes the expectation values of operators as described in Sec. 2.3.

3.4 Geometry versus field theory states

Given (3.10) we can explore the map between heavy 1/2-BPS states and ensembles that have a well-defined classical limit and the corresponding geometries. In the fermion representation described above, a sufficient condition for having a well-defined semi-classical limit is that the Young diagrams approach a fixed limiting shape with probability one in the large N limit. For the canonical ensemble, this limiting shape was given in (3.6) but for other states and ensembles different limit curves may arise in the large N limit. We will continue to denote those curves by $y(x)$ as in Sec. 3.2. They will describe the effective, coarse grained geometry corresponding to the states/ensembles.

To extract the geometry, we can use the identification (3.10) and work out the semiclassical limit of the phase space distributions of field theory states. However, for ensembles with limiting Young diagrams there is a shortcut. Using the fact that the phase space distribution should be rotationally invariant, and by matching energy \leftrightarrow conformal dimension, flux \leftrightarrow rank of the gauge group (=number of fermions), we get

$$N = \int dx = \int \frac{u(0; r^2)}{2\hbar} dr^2, \quad E = \int (x + y(x)) dx = \int \frac{r^2 u(0; r^2)}{4\hbar^2} dr^2 = \Delta. \quad (3.11)$$

The above equations should not just hold at the level of integrals but also at the level of integrands, so that

$$\frac{u(0; r^2)}{2\hbar} dr^2 = dx, \quad \frac{r^2 u(0; r^2)}{4\hbar^2} dr^2 = (y(x) + x) dx.$$

Combining these yields

$$y(x) + x = \frac{r^2}{2\hbar}$$

and taking derivatives with respect to x , we obtain the identification [23]

$$u(0; r^2) = \frac{1}{1 + y'}. \quad (3.12)$$

So given a Young diagram with a limit shape $y(x)$, one can associate to it a geometry generated by $u(0; r^2)$ according to (3.12).

For the limit curve of the canonical ensemble (3.6) the resulting phase space distribution is just that of a finite temperature Fermi gas system which was considered in [63]. Since u will lie between 0 and 1 the resulting geometry will have a null singularity. For the canonical ensemble of 1/2-BPS states with the limit curve (3.6), the explicit metric is not known. Since this would give the closest thing to a “1/2-BPS black hole,” in the sense that it would be the 1/2-BPS geometry with a given mass that has the most entropy, it would be interesting to know what it looks like.¹⁸ We can also ask what sort of ensemble gives rise to the known extremal, charged 1/2-BPS black hole [54], the so-called superstar. It was shown in [23] that this is an ensemble where the number of columns of the BPS Young diagram is held fixed in proportion to N as $N_c = \alpha N$ (i.e. where the maximum excitation energy in the fermionic description is held fixed). In this case the limiting diagram turns out to have a constant slope $y' = \alpha$, reproducing, via (3.12) and (3.7), the superstar geometry.

The above results showed that the overwhelming majority of 1/2-BPS basis states lie close to a certain “typical” configuration, which, in the semiclassical limit, corresponds to a singular geometry. What then, is the status of the many non-singular geometries ((3.7) with $u = 0, 1$) that have the same mass and charge as the “typical” configuration? We expect that almost all of these non-singular configurations will have string scale differences from the “typical” configuration, and after coarse-graining as appropriate to a semi-classical probe become indistinguishable from each other and from the singular effective description as an extremal black hole. This should lead to effective information loss in the semiclassical theory even though the underlying system is unitary. To study this further we can ask how semiclassical probes will respond to specific geometries derived from particular half-BPS states according to the correspondence (3.10).

3.5 Detecting states and semiclassical information loss

The 1/2-BPS fermion basis states described above are completely characterized by the individual fermion energies $\{e_1, \dots, e_N\}$. A gauge invariant set of variables containing the same data

¹⁸Notice that in our conventions $y' \geq 0$ and therefore the associated geometry generically has null singularities with $0 < u(0; r^2) < 1$; the unphysical cases with $u < 0$ or $u > 1$, which give rise to naked time-like singularities [60], do not appear.

are the moments

$$M_k = \sum_{i=1}^N e_i^k = \text{Tr}(H_N^k / \hbar^k) \quad ; \quad k = 0, \dots, N, \quad (3.13)$$

where H_N is the Hamiltonian acting on the N fermion Hilbert space with the zero point energy removed. The M_k are conserved charges of the system of fermions in a harmonic potential from which the individual energies e_i can be reconstructed. The basis of states with fixed fermion excitation energies that was described above consists of eigenstates of the moment operators. Above we showed that almost all heavy eigenstates of the M_k lie near a particular “typical” state. Nevertheless, an individual state of this kind can always be identified from the N eigenvalues of the M_k . Hence we can ask what 1/2-BPS spacetime corresponds to an individual joint eigenstate of all the M_k , and how an observer might measure the conserved eigenvalues of the M_k , thus identifying the underlying state.

A given eigenstate of the M_k corresponds to a specific Young diagram which can be translated to a corresponding geometry via the identification (3.10) between the phase space density and the function u that sources the LLM geometries (3.7). In [32] it was shown that in the semiclassical limit the k th multipole moment of the dual spacetime A_k is proportional to M_k , i.e. $A_k \propto M_k$. Thus, in the semiclassical limit, a 1/2-BPS basis state could be identified completely in spacetime if the angular moments of the spacetime can be measured.

First, observe that the semiclassical $\hbar \rightarrow 0$ with $\hbar N = \alpha$ fixed translates into gravity, using $l_p^4 \leftrightarrow \hbar$, as

$$\hbar N \leftrightarrow l_p^4 N \sim g_s l_s^4 N \sim L^4 = \alpha = \text{fixed} \quad ; \quad L \sim l_s (g_s N)^{1/4} \quad (3.14)$$

where l_s is the string length, g_s is the string coupling, and L is the length scale associated to the asymptotic $\text{AdS}_5 \times \text{S}^5$ spacetime using the standard AdS/CFT dictionary. Thus, the semiclassical limit that we have been using is the same as the standard limit in the AdS/CFT correspondence, namely $g_s \rightarrow 0, N \rightarrow \infty$ with L fixed. Now, following [32], one way of measuring the l^{th} multipole is to compute the $(2l)^{\text{th}}$ derivative of the metric functions or any suitable invariant constructed from them. Consider an apparatus of finite size λ that makes such a measurement. In order to compute the k^{th} derivative of a quantity within a region of size λ , the apparatus will have to make measurements at a scale λ/k . However, a semiclassical apparatus can only measure quantities over distances larger than the Planck length. Thus, the k^{th} derivative can only be measured if

$$\frac{\lambda}{k} > l_p = g_s^{1/4} l_s \quad (3.15)$$

Setting the size of the apparatus to be a fixed multiple of the AdS scale

$$\lambda = \gamma L, \quad (3.16)$$

this says that

$$k < \gamma N^{1/4} \quad (3.17)$$

for a derivative to be semiclassically measurable. In order to identify the underlying quantum state we have shown that $O(N)$ multipoles must be measured. Since $N^{1/4}/N \rightarrow 0$ as $N \rightarrow \infty$ we see that the semiclassical observer has access to a negligible fraction of the information needed to identify the quantum state. To make matters worse, in [32] it was also shown that in the ensemble of states with a fixed conformal dimension the standard deviation to mean ratio is

$$\frac{\sigma(M_k)}{\langle M_k \rangle} = \frac{k}{\sqrt{N(2k+1)}}. \quad (3.18)$$

This vanishes in the semiclassical limit for k that grow slower than N because of the typicality of heavy microstates that was discussed above. Thus, low moments are universal and thus do not differentiate states in the ensemble.

Thus, in the semiclassical limit the high moments are unmeasurable, while the low moments are universal, leading to effective loss of information.

3.6 Semiclassical limit, singularities and information loss

To study the semiclassical limit, it is convenient to follow [28] and place the N fermions of Sec. 3.1 in coherent states rather than in number eigenstates as we have been doing. Recall that a coherent state labelled by a parameter $\alpha \in \mathbb{C}$, is a Gaussian wavepacket in phase space localized around $\alpha = \frac{y_1 + i y_2}{\sqrt{2\hbar}}$. Coherent states are defined by

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \equiv \sum_{n=0}^{\infty} c_n(\alpha) |n\rangle. \quad (3.19)$$

These states are overcomplete, and in the semiclassical limit, a complete basis of coherent states can be thought of as inhabiting a lattice in which the unit cells have phase space area $2\pi\hbar$ [64, 65, 66].

A semiclassical observer measures the phase plane at an area scale $\delta A = 2\pi\hbar M \gg 2\pi\hbar$. Equivalently, in the dual spacetime the semiclassical observer makes measurements at scales much bigger than the Planck length. At this scale, the observer is only sensitive to a smooth, coarse grained phase space distribution $0 \leq 2\pi\hbar w(p, q) \equiv u(y_1, y_2) \leq 1$ which erases many details of the precise underlying precise microstates. We may view the region δA as consisting of $M = \delta A/2\pi\hbar$ lattice sites, a fraction $u_c = \hbar W_c$ of which are occupied by coherent states. Then the entropy of the local region δA is

$$S_K = \log \binom{M}{\hbar W_c M} \sim -M \log(\hbar W_c)^{\hbar W_c} (1 - \hbar W_c)^{1 - \hbar W_c} = -\frac{\delta A}{2\pi\hbar} \log u_c^{u_c} (1 - u_c)^{1 - u_c}. \quad (3.20)$$

The Stirling approximation used in (3.20) is valid when $\hbar W_c$ is reasonably far from 0 and 1.

For the total entropy this gives

$$S = \int dS = \int dA \left(\frac{dS}{dA} \right) \quad (3.21)$$

$$\frac{dS}{dA} = - \frac{u_c \log u_c + (1 - u_c) \log (1 - u_c)}{2\pi\hbar}. \quad (3.22)$$

Thinking about $u = 2\pi\hbar w$ as the probability of occupation of a site by a coherent state, this is simply Shannon’s formula for information in a probability distribution.¹⁹ It would be interesting to re-derive this result directly from the Young tableaux picture where the smooth limit shape encapsulates the classical observer’s ignorance of the underlying discrete tableau.

These facts imply that in the semiclassical limit the function $u(y_1, y_2)$ which completely defines a classical solution should effectively be defined on a lattice with each plaquette of area $\mathcal{O}(\hbar) \leftrightarrow \mathcal{O}(\ell_P^4)$, and take values of 0 or 1 in each site.²⁰ Likewise (3.22) should be interpreted as an expression for the entropy of arbitrary half-BPS asymptotically $\text{AdS}_5 \times S^5$ spacetimes. As an example it exactly reproduces the formula for the entropy of the typical states described in [23] that correspond in spacetime to the “superstar” geometry [54].

Note that the entropy vanishes if and only if u equals 0 or 1 everywhere. Following the correspondence (3.10) such states map into geometries that are non-singular. We learn that semiclassical half-BPS geometries that are smooth all have vanishing entropy; and the presence of singularities $0 < u < 1$ also implies that the spacetime carries an entropy. Thus, in this setting, entropy is a measure of ignorance of a part of the underlying state which is captured in classical gravity as a spacetime singularity.

3.7 Summary

In this section we discussed how the almost all half-BPS states of $\mathcal{N} = 4$ of a given conformal dimension lie close to a certain “typical” state. The semiclassical indistinguishability of these states leads to a universal description in AdS space as an extremal black hole. We further discussed how the presence of a singularity in a semiclassical geometry is directly related to a loss of information about underlying states that have the same effective description. Note that most states in the Hilbert space, when represented in either fermion excitation number basis or in the coherent state basis, will be random superpositions of the basis states. In [28] a criterion is given for determining which of these states can be effectively described in terms of a single geometry as opposed to a wavefunction over geometries.

¹⁹This result was arrived at independently by Masaki Shigemori in unpublished work.

²⁰Recall that in (3.7) y_1 and y_2 have units of length *squared*. This is why an ℓ_P^4 appears here.

4 $\text{AdS}_3 \times \text{S}^3$

In this section we are going to discuss the bound states of D1 and D5 branes in type II-B string theory compactified on²¹ $\text{T}^4 \times \text{S}^1$. These are 1/2-BPS states (preserving 8 supercharges) that describe a black hole without a classical horizon in 5 dimensions.²² However, we are going to work in 6 dimensions, explicitly keeping track of the S^1 . One of the reasons behind this decision is that in this way one gets solutions that are asymptotically $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ after taking a suitable decoupling limit. Thus we can employ the AdS/CFT machinery and benefit from the known properties of the dual two-dimensional conformal field theory.

As is well-known, the 1/2-BPS states in the CFT dual to the D1-D5 system can be identified with the states at level $L_0 = N_1 N_5$ in a system with $b_1 + b_3$ chiral fermions and $b_0 + b_2 + b_4$ chiral bosons, where $b_i = \dim H^i(M_4)$ for a compactification on $M_4 \times \text{S}^1$. Here N_1 and N_5 are the quantized number of D1 and D5 branes. Notice that this identification of 1/2 BPS states with a system of free bosons and fermions is only valid at the level of the Hilbert space, not at the level of correlation functions. Thus, we would ideally like to be able to find a detailed map between states/ensembles in this auxiliary theory of free bosons and fermions and half-BPS solutions of six-dimensional supergravity. In what follows we will describe such a map. We will first review the known supergravity solutions and their quantization, and then propose a map which is again based on the notion of phase space densities. We conclude this section by discussing various relevant examples.

4.1 The supergravity solution and its quantization

Starting with a fundamental string with transversal profile $\mathbf{F}(s) \subset \mathbb{R}^4$ then dualizing, one gets the following solutions [9, 8, 48], written in the string frame²³

$$ds^2 = \frac{1}{\sqrt{f_1 f_5}} [-(dt + A)^2 + (dy + B)^2] + \sqrt{f_1 f_5} d\mathbf{x}^2 + \sqrt{f_1/f_5} d\mathbf{z}^2$$

$$e^{2\Phi} = \frac{f_1}{f_5}, \quad C = \frac{1}{f_1} (dt + A) \wedge (dy + B) + \mathcal{C} \quad (4.1)$$

where y parametrizes a circle with coordinate radius R , z^i are coordinates on T^4 with coordinate volume V_4 , the Hodge star $*_4$ is defined with respect to the 4 dimensional non compact space

²¹The same story carries over to the case of $\text{K3} \times \text{S}^1$.

²²Strictly speaking the extremal black hole and the associated microstates are ground states in the Ramond sector of the theory, but by spectral flow they can be represented as 1/2-BPS states in the NS sector. Hence we will refer to them as 1/2-BPS states.

²³We are going to follow the conventions of [48].

spanned by x^i and

$$\begin{aligned} dB &= *_4 dA, \quad d\mathcal{C} = - *_4 df_5, \quad A = \frac{Q_5}{L} \int_0^L \frac{F'_i(s) ds}{|\mathbf{x} - \mathbf{F}(s)|^2} \\ f_5 &= 1 + \frac{Q_5}{L} \int_0^L \frac{ds}{|\mathbf{x} - \mathbf{F}(s)|^2}, \quad f_1 = 1 + \frac{Q_5}{L} \int_0^L \frac{|\mathbf{F}'(s)|^2 ds}{|\mathbf{x} - \mathbf{F}(s)|^2}. \end{aligned} \quad (4.2)$$

The solutions are asymptotically $\mathbb{R}^{1,4} \times S^1 \times T^4$. We can take a decoupling limit which simply amounts to erasing the 1 from the harmonic functions. The resulting metric will then be asymptotically $\text{AdS}_3 \times S^3 \times T^4$.

As mentioned above, the solutions are parametrized in terms of a closed curve

$$x_i = F_i(s), \quad 0 < s < L, \quad i = 1, \dots, 4. \quad (4.3)$$

In the sequel we are going to ignore oscillations in the T^4 direction as well as fermionic excitations; for a further discussion of these degrees of freedom see [67, 68]. The D1 (D5) charge Q_1 (Q_5) satisfy

$$L = \frac{2\pi Q_5}{R}, \quad Q_1 = \frac{Q_5}{L} \int_0^L |\mathbf{F}'(s)|^2 ds. \quad (4.4)$$

It turns out that the space of classical solutions has finite volume and therefore will yield a finite number of quantum states. To see this, one first starts by expanding \mathbf{F} in oscillators:

$$\mathbf{F}(s) = \mu \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}} \left(\mathbf{c}_k e^{i\frac{2\pi k}{L}s} + \mathbf{c}_k^\dagger e^{-i\frac{2\pi k}{L}s} \right) \quad (4.5)$$

where $\mu = \frac{g_s}{R\sqrt{V_4}}$. Then one computes the restriction of the Poisson bracket to the space of solutions (4.1) which turns out to be [47, 48]

$$[c_k^i, c_{k'}^{j\dagger}] = \delta^{ij} \delta_{kk'}. \quad (4.6)$$

After quantization, the relation between Q_1 and Q_5 reads

$$\left\langle \int_0^L : |\mathbf{F}'(s)|^2 : ds \right\rangle = \frac{(2\pi)^2}{L} \mu^2 N \quad (4.7)$$

where

$$N_1 = \frac{g_s}{V_4} Q_1, \quad Q_5 = g_s N_5, \quad N \equiv N_1 N_5 = \sum_{k=1}^{\infty} k \left\langle \mathbf{c}_k^\dagger \mathbf{c}_k \right\rangle. \quad (4.8)$$

N_1 , N_5 count the number of D1 and D5 branes respectively. The modes c_k^i become the creation and annihilation modes of four of the total of $b_0 + b_2 + b_4$ bosons; one can check that the four that appear are precisely the ones associated to the $H^{(0,0)}(M)$, $H^{(2,0)}(M)$, $H^{(0,2)}(M)$, $H^{(2,2)}(M)$ cohomology. Finally, notice that the number of states and hence the entropy can easily be extracted from the known partition functions of chiral bosons and fermions.

4.2 Geometries from coherent states

The Hilbert space is spanned by

$$|\psi\rangle = \prod_{i=1}^4 \prod_{k=1}^{\infty} (c_k^{i\dagger})^{N_{k_i}} |0\rangle, \quad \sum k N_{k_i} = N \quad (4.9)$$

Given a state, or more generically a density matrix in the CFT

$$\rho = \sum_{ij} c_{ij} |\psi_i\rangle \langle \psi_j| \quad (4.10)$$

we wish to associate to it a density on phase space. The phase space is given by classical curves which we will parametrize as (note that d and \bar{d} are now complex numbers, not operators)

$$\mathbf{F}(s) = \mu \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}} \left(\mathbf{d}_k e^{i\frac{2\pi k}{L}s} + \bar{\mathbf{d}}_k e^{-i\frac{2\pi k}{L}s} \right) \quad (4.11)$$

and which obey the classical constraint (4.4).

We now propose to associate to a density matrix of the form (4.10) a phase space density (compare to the general discussion in section 2.1) of the form [25]

$$f(d, \bar{d}) = \sum_i \frac{\langle 0 | e^{\mathbf{d}_k \mathbf{c}_k} |\psi_i\rangle \langle \psi_i| e^{\bar{\mathbf{d}}_k \mathbf{c}_k^\dagger} |0\rangle}{\langle 0 | e^{\mathbf{d}_k \mathbf{c}_k} e^{\bar{\mathbf{d}}_k \mathbf{c}_k^\dagger} |0\rangle}. \quad (4.12)$$

The distribution corresponding to a generic state $|\psi\rangle = \prod_{k=1}^{\infty} (c_k^{i\dagger})^{N_{k_i}} |0\rangle$ can be easily computed

$$f(d, \bar{d}) = \prod_{k,i} (d_k^i \bar{d}_k^i)^{N_{k_i}} e^{-d_k^i \bar{d}_k^i}. \quad (4.13)$$

Notice that our phase space density (4.12), as written, is a function on a somewhat larger phase space as d, \bar{d} do not have to obey (4.4). To cure this discrepancy we are going to include an extra factor $\exp(-\beta \hat{N})$ in the calculations, where we choose β such that the expectation value of \hat{N} is precisely N . This is just like passing from a microcanonical ensemble to a canonical one, and for many purposes this is probably a very good approximation. For a thorough discussion of this point see [25].

To further motivate (4.12) we notice that it associates to a coherent state a density which is a Gaussian centered around a classical curve, in perfect agreement with the usual philosophy that coherent states are the most classical states. It is then also clear that given a classical curve (4.11) we wish to associate to it the density matrix

$$\rho = P_N e^{\mathbf{d}_k \mathbf{c}_k} |0\rangle \langle 0| e^{\bar{\mathbf{d}}_k \mathbf{c}_k^\dagger} P_N \quad (4.14)$$

where P_N is the projector onto the actual Hilbert space of states of energy N as defined in (4.9). Because of this projector, the phase space density associated to a classical curve is not

exactly a Gaussian centered around the classical curve but there are some corrections due to the finite N projections. Obviously, these corrections will vanish in the $N \rightarrow \infty$ limit.

The density (4.12) has the property that for any function $g(d, \bar{d})$

$$\int \int_{d, \bar{d}} f(d, \bar{d}) g(d, \bar{d}) = \sum_i \langle \psi_i | : g(c, c^\dagger) :_A | \psi_i \rangle \quad (4.15)$$

where $: g(c, c^\dagger) :_A$ is the anti-normal ordered operator associated to $g(c, c^\dagger)$, and $\int_{d, \bar{d}}$ is an integral over all variables d_i . Since the theory behaves like a $1 + 1$ dimensional field theory the natural thing to do is to calculate expectation values of normal ordered operators in order to avoid infinite normal ordering contributions. Besides, everything we do is limited by the fact that our analysis is in classical gravity and therefore can at best be valid up to quantum corrections. As a result a further modification to our proposal will be to redefine $g(d, \bar{d})$ by subtracting the anti-normal ordering effects.

Since the harmonic functions appearing in (4.2) can be arbitrarily superposed, we finally propose to associate to (4.10) the geometry

$$\begin{aligned} f_5 &= 1 + \frac{Q_5}{L} \mathcal{N} \int_0^L \int_{d, \bar{d}} \frac{f(d, \bar{d}) ds}{|\mathbf{x} - \mathbf{F}(s)|^2} \\ f_1 &= 1 + \frac{Q_5}{L} \mathcal{N} \int_0^L \int_{d, \bar{d}} \frac{f(d, \bar{d}) |\mathbf{F}'(s)|^2 ds}{|\mathbf{x} - \mathbf{F}(s)|^2} \\ A^i &= \frac{Q_5}{L} \mathcal{N} \int_0^L \int_{d, \bar{d}} \frac{f(d, \bar{d}) \mathbf{F}'_i(s) ds}{|\mathbf{x} - \mathbf{F}(s)|^2} \end{aligned} \quad (4.16)$$

with the normalization constant

$$\mathcal{N}^{-1} = \int_{d, \bar{d}} f(d, \bar{d}) \quad (4.17)$$

In [8] it was shown that the geometries corresponding to a classical curve are regular provided $|\mathbf{F}'(s)|$ is different from 0 and the curve is not self intersecting. In our setup we sum over continuous families of curves with some weighing factor which can introduce singularities. We expect these singularities to be rather mild, certainly for semiclassical density matrices, and in addition in various examples the averages will turn out to be completely smooth anyway (see section 4.3). Another point worth mentioning is that the average will no longer solve the vacuum type IIB equations of motion, instead a small source will appear on the right hand side of the equations. Since these sources are subleading in the $1/N$ expansion and vanish in the classical limit, they are in a regime where classical gravity is not valid and they may well be cancelled by higher order contributions to the equations of motion. To have an idea about these sources let us study the circular profile.

We consider the following profile

$$F^1(s) = a \cos \frac{2\pi k}{L} s, \quad F^2(s) = a \sin \frac{2\pi k}{L} s, \quad F^3(s) = F^4(s) = 0 \quad (4.18)$$

which describes a circular curve winding k times around the origin in the 12-plane. In order to simplify our discussion, we focus on the simplest harmonic function f_5 . In order to evaluate the various integrals it will be convenient to Fourier transform the x -dependence. Using

$$\frac{1}{|\mathbf{x}|^2} = \frac{1}{4\pi^2} \int d^4u \frac{e^{i\mathbf{u}\cdot\mathbf{x}}}{|\mathbf{u}|^2} \quad (4.19)$$

Classically $\square f_5$ is a delta function with a source at the location of the classical curve, to be precise

$$\square f_5 = -4\pi Q_5 \delta(x_1^2 + x_2^2 - a^2) \delta(x_3) \delta(x_4). \quad (4.20)$$

Now in the quantum theory, we associate to the classical circular curve (4.18) the density matrix (4.14) and subsequently the phase space density (4.12). Working this out we find out that

$$f(d, \bar{d}) = ((d_k^1 + id_k^2)(\bar{d}_k^1 - id_k^2))^{N/k} e^{-\sum_{l,i} d_l^i \bar{d}_l^i}. \quad (4.21)$$

We have ignored the delta function coming from the projection here and expect (4.21) to be valid for large values of N/k . It is therefore better thought of as a semiclassical profile rather than the full quantum profile.

According to (4.16) the harmonic function f_5 is now given by

$$f_5 = 1 + \frac{Q_5}{4\pi^2} \mathcal{N} \int_0^L ds \int_{d, \bar{d}} f(d, \bar{d}) \int d^4u \frac{1}{|\mathbf{u}|^2} e^{i\mathbf{u}\cdot(\mathbf{x}-\mathbf{F}(s)) + \sum_l \frac{u^2 \mu^2}{2l}} \quad (4.22)$$

where we have used (4.19) and the constant $\sum_l \frac{u^2 \mu^2}{2l}$ appears due to the fact that we want to compute a normal ordered quantity instead of an anti-normal ordered one. The function $\mathbf{F}(s)$ depends on an infinite set of complex oscillators d_l^i . It can be easily seen that the contribution for the oscillators different from d_k^1 and d_k^2 cancels exactly against the normal ordering constant $u^2 \mu^2 / 2l$ mentioned above.

So $\square f_5$ for this case reads

$$\square f_5 = -4\pi Q_5 \delta(x_3) \delta(x_4) A(x_1, x_2) \quad (4.23)$$

$$A(x_1, x_2) = \int_0^\infty d\rho \rho J_0(\sqrt{x_1^2 + x_2^2} \rho) L_{N/k} \left(\frac{a^2 \rho^2}{4N/k} \right) \quad (4.24)$$

Until here we have not used any approximation. Using the identity

$$L_N(x) = \frac{e^x}{N!} \int_0^\infty e^{-t} t^N J_0(2\sqrt{tx}) dt$$

and approximating $\exp(\frac{a^2 \rho^2}{4N/k}) \approx 1$ one obtains

$$A(x_1, x_2) = \frac{e^{-N/k} r^2/a^2 (N/k) r^2/a^2)^{N/k}}{(N/k - 1)! a^2} \quad (4.25)$$

with $r^2 = x_1^2 + x_2^2$. In the limit $N/k \rightarrow \infty$ $A(x_1, x_2)$ approaches $\frac{\delta(r^2/a^2 - 1)}{a^2}$ and the classical and quantum results agree. For large N/k $A(x_1, x_2)$ is approximately a Gaussian around $r^2 \approx a^2$ and width $1/\sqrt{N/k}$, indeed, using Stirling's formula

$$A(x_1, x_2) \approx \frac{\sqrt{N/k}}{\sqrt{2\pi}} e^{-N/k(r^2/a^2 - 1)} (r^2/a^2)^{N/k} \quad (4.26)$$

So the geometry corresponds to a solution of the equations of motion in presence of smeared sources. The width of the smeared source goes to zero in the limit $N/k \rightarrow \infty$, as expected.

4.3 Ensembles

In the following we consider the geometry of some ensembles of interest.

4.3.1 $M = 0$ BTZ

The corresponding ensemble is characterized by the following density matrix ²⁴

$$\rho = \sum_{N_k, \tilde{N}_k} \frac{|N_k\rangle\langle N_k| e^{-\beta \hat{N}} |\tilde{N}_k\rangle\langle \tilde{N}_k|}{\text{Tr} e^{-\beta \hat{N}}} \quad (4.27)$$

where $|N_k\rangle$ is a generic state labelled by collective indices N_k

$$|N_k\rangle = \prod_k \frac{1}{\sqrt{N_k!}} (c_k^\dagger)^{N_k} |0\rangle$$

and we have chosen a normalization so that $\langle N_k | \tilde{N}_k \rangle = \delta_{N_k, \tilde{N}_k}$. The value of the potential β has to be adjusted such that $\langle \hat{N} \rangle = N$. It is clear that

$$\rho = \prod_n \rho_k, \quad \rho_k = (1 - e^{-k\beta}) \sum_{n=0}^{\infty} e^{-nk\beta} |k, n\rangle\langle k, n| \quad (4.28)$$

with $|k, n\rangle = \frac{1}{\sqrt{n!}} (c_k^\dagger)^n |0\rangle$. Then the full distribution will simply be the product $f(d, \bar{d}) = \prod_k f_{d_k, \bar{d}_k}^{(k)}$ with

$$f_{d_k, \bar{d}_k}^{(k)} = (1 - e^{-k\beta}) e^{-d_k \bar{d}_k} \sum_{n=0}^{\infty} \frac{e^{-nk\beta}}{n!} (d_k \bar{d}_k)^n = (1 - e^{-k\beta}) \exp(-(1 - e^{-k\beta}) d_k \bar{d}_k). \quad (4.29)$$

The needed harmonic functions 4.2 are deduced from the following generating function

$$f_v = \frac{Q_5}{4\pi^2 L} \mathcal{N} \int d^4 u \int_0^L dr \int_{d, \bar{d}} f(d, \bar{d}) \frac{e^{\sum_k \left(\frac{|\mathbf{u}|^2 \mu^2}{2k} - \frac{2\pi^2 k \mu^2 |\mathbf{v}|^2}{L^2} \right)}}{|\mathbf{u}|^2} e^{i\mathbf{u} \cdot (\mathbf{x} - \mathbf{F}(r)) + i\mathbf{v} \cdot \mathbf{F}'(r)} \quad (4.30)$$

²⁴We are going to ignore the i -index in some equations where it does not play any role. We hope that this will not create any confusion.

which gives

$$f_5 = Q_5 \frac{1 - e^{-\frac{3\beta}{\pi^2 \mu^2} x^2}}{x^2}, \quad f_1 = Q_1 \frac{1 - e^{-\frac{3\beta}{\pi^2 \mu^2} x^2}}{x^2}$$

$$A_i = 0, \quad \beta \approx \pi \sqrt{\frac{2}{3N}}. \quad (4.31)$$

A final comment is in order. The geometry obtained differs from the *classical* $M = 0$ BTZ black hole by an exponential piece. Following [69, 4] we could put a stretched horizon at the point where this exponential factor becomes of order one, so that the metric deviates significantly from the classical $M = 0$ BTZ solution. Thus, using this criterion we find for the radius of the stretched horizon²⁵

$$r_0 \approx \frac{\mu}{\beta^{1/2}} \quad (4.32)$$

with corresponding entropy proportional to $N^{3/4}$. This exceeds the entropy of the mixed state from which the geometry was obtained; the latter grows as $N^{1/2}$. This does not contradict any known laws of physics, and in addition we should remember that the notion of stretched horizon depends on the choice of observer. It is quite likely that for a suitable choice of observer the entropy of the stretched horizon agrees with the entropy obtained from the dual CFT. For a further discussion of this point see [25, 70].

4.3.2 The small black ring

In this section we consider a slightly more complicated example, namely an ensemble consisting of a condensate of J oscillators of level q plus a ensemble of effective level $N - qJ$. As argued in [71, 29, 72, 24] such an ensemble should describe (in a certain region of parameter space) a small black ring of angular momentum J and dipole (or Kaluza-Klein) charge q .

Using the techniques developed in the previous sections we can compute the generating harmonic function for this case as well and we find

$$f_v = Q_5 L_J \left(\frac{\mu^2}{4q} \left[\left(\frac{2\pi q}{L} v_2 + i\partial_1 \right)^2 + \left(\frac{2\pi q}{L} v_1 - i\partial_2 \right)^2 \right] \right) e^{-\frac{\mu^2 \pi^2 |v|^2}{2L^2} (N - qJ)} \frac{1 - e^{-\frac{2|\mathbf{x}|^2}{\mu^2 D}}}{|\mathbf{x}|^2} \quad (4.33)$$

where $D = \pi \sqrt{2/3} (N - qJ)^{1/2}$ so that the geometry is purely expressed in terms of the macroscopic quantities N , J and q .

We would like to make contact between this geometry and the geometry corresponding to small black rings studied in [24]. As we will see, in the limit of large quantum numbers both geometries reproduce the same one point functions.

²⁵The same value is obtained if we compute the average size of the curve in \mathbb{R}^4 , $r_0^2 \approx \langle |F|^2 \rangle$.

In order to see this, first note that the exponential factor $e^{-\frac{2|\mathbf{x}|^2}{\mu^2 D}}$ will not contribute (as it vanishes faster than any power at asymptotic infinity). Secondly one has the formal expansion

$$L_J \left(\frac{\mu^2}{4q} \mathcal{O} \right) = J_0 \left(\mu \sqrt{\frac{J}{q}} \mathcal{O}^{1/2} \right) + \dots \quad (4.34)$$

In order to estimate the validity of this approximation we can think of \mathcal{O} as being proportional to $1/|\mathbf{x}|^2$. On the other hand $\mu \sqrt{J/q}$ can be roughly interpreted as the radius of the black ring (see [73, 24], where this parameter is called R). Hence the approximation is valid for large values of J at a fixed distance compared to the radius of the ring.

Using the above approximations it is straightforward to compute the harmonic functions

$$f_5 = \frac{Q_5}{r^2 + \mu^2 \frac{J}{q} \cos \theta}, \quad f_1 = \frac{Q_1}{r^2 + \mu^2 \frac{J}{q} \cos \theta} \quad (4.35)$$

where we have used the following coordinate system

$$\begin{aligned} x_1 &= (r^2 + a^2)^{1/2} \sin \theta \cos \varphi, & x_2 &= (r^2 + a^2)^{1/2} \sin \theta \sin \varphi \\ x_3 &= r \cos \theta \cos \psi, & x_4 &= r \cos \theta \sin \psi. \end{aligned} \quad (4.36)$$

Hence in this approximation the geometry reduces exactly to that of the small black ring studied in [24].

4.3.3 Generic ensemble and the no-hair theorem

In the following we consider a generic ensemble, where each oscillator c_{k^i} is occupied thermally with a temperature β_{k^i} . We further will assume that β_{k^\pm} for the directions 1, 2 is equal to β_{k^\pm} for the directions 3, 4. Restricting to, say, directions 1, 2 we are led to consider the following distribution

$$f(d, \bar{d}) = \exp \left(-(1 - e^{-\beta_{k^+}}) d_k^+ \bar{d}_k^+ - (1 - e^{-\beta_{k^-}}) d_k^- \bar{d}_k^- \right). \quad (4.37)$$

Following the same steps as for the case of the small black ring we obtain

$$f_5 = Q_5 \frac{1 - e^{-\frac{2|\mathbf{x}|^2}{\mu^2 D}}}{|\mathbf{x}|^2} \quad (4.38)$$

$$f_1 = Q_1 \left(\frac{1 - e^{-\frac{2|\mathbf{x}|^2}{\mu^2 D}}}{|\mathbf{x}|^2} - \frac{J^2}{4N\mu^4 D^2} e^{-\frac{2|\mathbf{x}|^2}{\mu^2 D}} \right) \quad (4.39)$$

$$A = \frac{\mu^2 J}{2} \left(2 \frac{e^{-\frac{2|\mathbf{x}|^2}{\mu^2 D}}}{\mu^2 D} - \frac{1 - e^{-\frac{2|\mathbf{x}|^2}{\mu^2 D}}}{|\mathbf{x}|^2} \right) (\cos^2 \theta d\phi + \sin^2 \theta d\psi) \quad (4.40)$$

where $(|\mathbf{x}|, \theta, \phi, \psi)$ are standard spherical coordinates on \mathbb{R}^4 .

We see that, rather surprisingly, the geometry depends only on a few quantum numbers N, J and D which are given in terms of the temperatures by

$$N = 2 \sum_k k \left(\frac{e^{-\beta_{k+}}}{1 - e^{-\beta_{k+}}} + \frac{e^{-\beta_{k-}}}{1 - e^{-\beta_{k-}}} \right) \quad (4.41)$$

$$J = 2 \sum_k \left(\frac{e^{-\beta_{k+}}}{1 - e^{-\beta_{k+}}} - \frac{e^{-\beta_{k-}}}{1 - e^{-\beta_{k-}}} \right) \quad (4.42)$$

$$D = 2 \sum_k \frac{1}{k} \left(\frac{e^{-\beta_{k+}}}{1 - e^{-\beta_{k+}}} + \frac{e^{-\beta_{k-}}}{1 - e^{-\beta_{k-}}} \right). \quad (4.43)$$

As a result, the information carried by the geometry is much less than that carried by the ensemble of microstates. In fact, only N and J are visible at infinity while D sets the size of the “core” of the geometry. We also find that D is precisely the expectation value of the dipole operator introduced in [24]. Its presence in the density matrix is supported by an analysis of the first law of thermodynamics [74]. It is a non-conserved charge which makes its extension to interacting theories an interesting open problem.

We interpret the above remark as a manifestation of the no-hair theorem for black holes. The derivation in this section assumes that the temperatures are all sufficiently large. By tuning the temperatures, it is possible to condense one (like in the small black ring case) or more oscillators. If this happens, we should perform a more elaborate analysis, and we expect that the dual geometrical description²⁶ corresponds to concentric small black rings. In this case the configuration will depend on more quantum numbers than just N, J, D , in particular we will find solutions where the small black rings carry arbitrary dipole charge. Thus, once we try to put hair on the small black hole by tuning chemical potentials appropriately, we instead find a phase transition to a configuration of concentric small black rings, each of which still is characterized by just a few quantum numbers.

4.4 BTZ $M = 0$ as an effective geometry

Above we discussed how the half-BPS states of the D1-D5 CFT and ensembles of these states can be mapped to specific asymptotically AdS_3 geometries. In this section we will argue, following [29] that the *typical* state in this sector responds to probes as if it were a BTZ $M = 0$ black hole in spacetime. This approach is complementary to Sec. 4.3.1 where it was shown how BTZ arises as the effective, coarse-grained geometry of an ensemble of D1-D5 states. It would be interesting check more carefully if the two-point function computed below can be somehow sensitive to corrections from the stretched horizon described in Sec. 4.3.1. For concreteness we will restrict our analysis here to T^4 compactifications.

²⁶It is not difficult to see that the harmonic functions now will take the form of multiple Laguerre polynomials with differential operator arguments acting on the generating harmonic function of the $M = 0$ BTZ solution.

The CFT dual to $\text{AdS}_3 \times S^3 \times T^4$ is an $\mathcal{N} = (4, 4)$ supersymmetric sigma model whose target space is the symmetric product $\mathcal{M}_0 = (T^4)^N / S_N$, where S_N is the permutation group of order N . Here we set

$$N \equiv N_1 N_5. \quad (4.44)$$

More precisely, \mathcal{M}_0 is the so-called orbifold point in a family of CFTs which are regained by turning on certain marginal deformations of the sigma model on \mathcal{M}_0 . At the orbifold point the CFT becomes free. The CFT has a collection of twist fields σ_n , which cyclically permute $n \leq N$ copies of the CFT on a single T^4 . One can think of these operators as creating winding sectors of the worldsheet that wind over the different copies of the torus.

After spectral flow from the NS sector to the Ramond sector, all of the 1/2-BPS states discussed above become Ramond sector ground states, and we will choose to describe the system in the latter language. A general Ramond sector ground state is constructed by multiplying together elementary bosonic and fermionic twist operators to achieve a total twist of $N = N_1 N_5$:

$$\begin{aligned} \sigma &= \prod_{n,\mu} (\sigma_n^\mu)^{N_{n\mu}} (\tau_n^\mu)^{N'_{n\mu}}, \\ \sum_{n,\mu} n(N_{n\mu} + N'_{n\mu}) &= N, \quad N_{n\mu} = 0, 1, 2, \dots, \quad N'_{n\mu} = 0, 1. \end{aligned} \quad (4.45)$$

Here σ_n^μ and τ_n^μ are the constituent elementary twist operators, and $\mu = 1 \dots 8$ labels their possible polarizations. This is like 8 bosonic and 8 fermionic oscillators. Secs. 4.1, 4.2, 4.3 above constructed states out of just 4 bosonic oscillators because a convenient subset of the states was examined which lacked oscillations in the T^4 and fermionic excitations. Here we are considering the full Hilbert space of states. The Appendix in [29] gives a detailed description of the construction of the twist operators and computations using them. For our immediate purposes, the relevant point is that the integers

$$\{N_{n\mu}, N'_{n\mu}\} \quad (4.46)$$

uniquely specify a Ramond ground state.

When the total twist length $N = \sum_{n,\mu} n(N_{n\mu} + N'_{n\mu})$ is very large, there are a macroscopic number ($\sim e^{2\sqrt{2}\pi\sqrt{N}}$) of Ramond ground states. In such a situation, most of those $e^{2\sqrt{2}\pi\sqrt{N}}$ microstates will have a twist distribution $\{N_{n\mu}, N'_{n\mu}\}$ that lies very close to a certain “typical” distribution. In the large N limit, the difference among individual distributions is small. Roughly, statistical mechanics says that $\langle (\Delta N_{n\mu})^2 \rangle \sim N_{n\mu}$, thus $\frac{\langle (\Delta N_{n\mu})^2 \rangle^{1/2}}{N_{n\mu}} \sim (N_{n\mu})^{-1/2} \rightarrow 0$ as $N_{n\mu} \rightarrow \infty$. Thus, although correlation functions computed in individual microstates depend on the precise form of the microstate distribution $\{N_{n\mu}, N'_{n\mu}\}$, for almost all microstates the generic responses should deviate by small amounts from the results for the typical state. This will be the basis for the emergence of an effective black hole description of typical Ramond ground states.

Let us first consider the ensemble of all the Ramond ground states (4.45) with equal statistical weight. The canonical partition function is

$$Z(\beta) = \text{Tr}[e^{-\beta N}] = \prod_{n=1}^{\infty} \frac{(1+q^n)^8}{(1-q^n)^8} = \left[\frac{\vartheta_2(0|\tau)}{2\eta(\tau)^3} \right]^4, \quad q = e^{2\pi i \tau} = e^{-\beta}. \quad (4.47)$$

Using the modular property of the theta function,

$$Z(\beta) = \left[\frac{\beta}{4\pi} \frac{\vartheta_4(0|\frac{1}{\tau})}{\eta(-\frac{1}{\tau})^3} \right]^4 \sim e^{2\pi^2/\beta} \quad (\beta \ll 1). \quad (4.48)$$

The relation between “energy” N and temperature β is

$$N = \left\langle \sum_{n=1}^{\infty} \sum_{\mu} n(N_{n\mu} + N'_{n\mu}) \right\rangle = -\frac{\partial}{\partial \beta} \ln Z(\beta) \simeq \frac{2\pi^2}{\beta^2}. \quad (4.49)$$

Since all twist operators are independent, the average distribution $\{N_{n\mu}, N'_{n\mu}\}$ is given by the Bose–Einstein and Fermi–Dirac distribution, respectively:

$$N_{n\mu} = \frac{1}{e^{\beta n} - 1}, \quad N'_{n\mu} = \frac{1}{e^{\beta n} + 1}, \quad N_n = \sum_{\mu} (N_{n\mu} + N'_{n\mu}) = \frac{8}{\sinh \beta n}. \quad (4.50)$$

For large N , the typical states of our ensemble have a distribution almost identical to (4.50). We will call the distribution (4.50) the “representative” distribution.

For simplicity, we will compute the 2-point functions of non-twist “probe” operators \mathcal{A} in states created by general twist operators. \mathcal{A} can be written as a sum over copies of the CFT:

$$\mathcal{A} = \frac{1}{\sqrt{N}} \sum_{A=1}^N \mathcal{A}_A \quad (4.51)$$

where \mathcal{A}_A is a *non-twist* operator that lives in the A -th copy. For example, we can take

$$\mathcal{A}_A = \partial X_A^a(z) \bar{\partial} X_A^b(\bar{z}), \quad (4.52)$$

which corresponds to a fluctuation of the metric in the internal T^4 direction. Although, such non-twist operators are only a subset of the operators that correspond to spacetime excitations, we will restrict ourselves to them because their correlation functions are much easier to compute than those of twist operators, and because they will be sufficient to demonstrate that an effective geometry emerges in the $N \rightarrow \infty$ limit.

Given a general Ramond ground state σ (4.45) we are interested in computing

$$\langle \sigma^\dagger \mathcal{A}^\dagger \mathcal{A} \sigma \rangle \quad (4.53)$$

The key result, demonstrated in the Appendix of [29], is that for non-twist operators at the orbifold point in the CFT such correlation functions decompose into independent contributions from the constituent twist operators in (4.45). Using this, it is shown in [29] that for a bosonic \mathcal{A} , we obtain

$$\langle \mathcal{A}(w_1) \mathcal{A}(w_2) \rangle_\Sigma = \frac{1}{N} \sum_n n N_n \sum_{k=0}^{n-1} \frac{C}{[2n \sin(\frac{w-2\pi k}{2n})]^{2h} [2n \sin(\frac{\bar{w}-2\pi k}{2n})]^{2\tilde{h}}}, \quad (4.54)$$

where

$$N_n \equiv \sum_\mu (N_{n\mu} + N'_{n\mu}). \quad (4.55)$$

Here $w = \phi - t/L$ and $\bar{w} = \phi + t/L$ with L being the AdS scale are lightcone coordinates in boundary CFT.

Let us study the relative size of the contributions to this from terms with different n . The contributions come multiplied by nN_n , which is $\frac{8n}{\sinh \beta n}$ for the typical microstates with $J = 0$ (Eq. (4.50)). Because of the suppression by the $\sinh \beta n$, the values of n that make substantial contributions to the correlation function (4.54) are $n \lesssim 1/\beta \sim \sqrt{N}$. Thus there are $O(\sqrt{N})$ twists that make a significant contribution. Now observe that for any $\gamma < 1/2$, the number of twists with $n \lesssim N^\gamma$ is parametrically smaller than \sqrt{N} . Indeed, the ratio vanishes as $N \rightarrow \infty$. In this sense we can say that in the $N \rightarrow \infty$ limit, (4.54) is dominated by twists scaling as $n \sim \sqrt{N}$.

Next, for any $n \geq 1$, when $t \ll n$ we can approximate the sum on k as

$$\sum_{k=0}^{n-1} \frac{1}{[2n \sin(\frac{w-2\pi k}{2n})]^{2h} [2n \sin(\frac{\bar{w}-2\pi k}{2n})]^{2\tilde{h}}} \approx \sum_{k=-\infty}^{\infty} \frac{1}{(w-2\pi k)^{2h} (\bar{w}-2\pi k)^{2\tilde{h}}} \quad (t \ll n),$$

where we assumed $h + \tilde{h} = \text{even}$. Putting the above statements together, we arrive at the following conclusion: for sufficiently early times

$$t \ll t_c = \mathcal{O}(\sqrt{N}), \quad (4.56)$$

the correlation function (4.54) can be approximated by

$$\begin{aligned} \langle \mathcal{A}(w_1) \mathcal{A}(w_2) \rangle_\Sigma &\approx \frac{1}{N} \sum_n n N_n \sum_{k=-\infty}^{\infty} \frac{C}{(w-2\pi k)^{2h} (\bar{w}-2\pi k)^{2\tilde{h}}} \\ &= \sum_{k=-\infty}^{\infty} \frac{C}{(w-2\pi k)^{2h} (\bar{w}-2\pi k)^{2\tilde{h}}}. \end{aligned} \quad (4.57)$$

This turns out to be *precisely* the correlation function computed in the $M = 0$ BTZ black hole background [29]. Therefore, in the orbifold CFT approximation, the emergent effective

geometry of the D1-D5 system is the $M = 0$ BTZ black hole geometry. The description in terms of this effective geometry is valid until $t \sim t_c \sim O(S)$. Here e^S is the statistical degeneracy of the Ramond ground states which goes to infinity in the semiclassical limit where $N \rightarrow \infty$.

Notice that in (4.57) the sum over the twists n factors out. Thus, for $t < t_c$ we are showing that the correlation function is largely independent of the detailed microscopic distribution of twists. It is this universal response that reproduces the physics of the $M = 0$ BTZ black hole. After $t \sim t_c$, the approximation (4.57) breaks down, and the correlation function starts to show random-looking, quasi-periodic behavior (see the figures in [29]). The form of the correlation function in this regime will depend on the precise form of the individual microstate, no matter how close it is to the representative state (4.50).

Note that the $M = 0$ BTZ black hole yields a correlator which decays to zero at large times as $1/t^2$. By contrast, the microstate correlators exhibit quasi-periodic fluctuations around a nonzero mean value. Numerical analysis indicates that this mean value scales as $\frac{1}{\sqrt{N}}$ for $h = \tilde{h} = 1$. For an ordinary finite size, finite temperature system, one expects the mean value to be of order e^{-cS} where S is the entropy and c is of order 1. This behavior arises because typical interactions can explore the entire phase space of the system. The fact that we observe power law rather than exponential behavior is partly a result of working in the free orbifold limit of the CFT and probing the system with only non-twist operators, so that the full space of states does not come into play. It is also worth noting that the generic state in the D1-D5 Hilbert space is a random *superposition* of basis states of the form (4.45). Then the arguments of Sec. 2.5 and [31] show that over the entire Hilbert space the variance of observables will be suppressed exponentially in the entropy.

Finally, a finite N microstate correlator will exhibit exact periodicity in time because only a finite number of frequencies appear in the Fourier expansion. The frequencies are $\omega_n = \frac{n}{N}$, $n = 1, 2, \dots, N$. Let $L(N)$ denote the least common multiple of $(1, 2, \dots, N)$. The correlator is then periodic with period $\Delta t = 2\pi N L(N)$. The large N behavior of $L(N)$ is $L(N) \sim e^N$, and therefore

$$\Delta t \sim N e^N . \quad (4.58)$$

Our correlators have been computed in the canonical ensemble in which the summation over n extends past N up to infinity, and so we will not see this exact periodicity. On the other hand, due to the exponential suppression of the distribution function N_n the deviation from exact periodicity is tiny for large N . As was argued above, and as can be confirmed numerically, one finds that for large N the large time behavior of the correlator is unaffected if we truncate the sum over n at $n_{\max} = c\sqrt{N}$, for c of order unity. Taking this into account, we see that our correlators will exhibit approximate periodicity with period

$$\Delta t \sim e^{c\sqrt{N}} = e^{\tilde{c}S} , \quad (4.59)$$

where $S = 2\pi\sqrt{2}\sqrt{N}$ is the entropy. This timescale is the so-called Poincaré recurrence time, over which generic finite size thermal systems are expected to exhibit approximate periodicity.

4.5 Summary

In this section we described how appropriately coherent BPS states and density matrices of the D1-D5 string can be mapped to specific geometries in asymptotically AdS_3 spacetimes. While specific coherent microstates mapped to particular horizon-free geometries, the thermal density matrices mapped to the BTZ $M=0$ black hole and the small black ring. The techniques for producing this mapping were in analogy to the phase methods used in the previous section to develop the relation between half-BPS geometries of AdS_5 and the dual Yang-Mills theory. We then showed that the *typical* BPS microstates of the D1-D5 string, reacts to probes in such a way that their effective description, on timescales that go to infinity in the semiclassical limit, is as a BTZ black hole. This is despite the fact that the individual coherent microstates can be mapped as described above to horizon-free BPS geometries. Note that the latter results were demonstrated in the orbifold limit of the D1-D5 CFT where the theory is actually free. This is possible because the essence of the problem of the emergence of black-hole like behavior is not the presence of interactions per se, but rather the enormous underlying degeneracy.

5 $\text{AdS}_3 \times \text{S}^2$

Although the extremal $D1$ - $D5$ system has proved a fertile example to test the idea that black holes are simply effective geometries, the extremal solution has a horizon coincident with the singularity and thus a finite horizon area only (possibly) emerges when higher derivative corrections are included. Thus this example has some special features that do not generalize. It would be desirable to be able to study a similar scenario for a system where the total charge corresponds to a black hole with a macroscopic horizon (i.e. a three charge black hole). Such black holes are $1/8$ BPS solutions in the full string theory or can emerge as $1/2$ BPS solutions of $\mathcal{N} = 2$ four dimensional or $\mathcal{N} = 1$ five dimensional supergravity (i.e. string or M -theory reduced on a Calabi-Yau).

5.1 Solution Spaces

The general multi-centered BPS solutions of generic $\mathcal{N} = 2$ supergravity theories in four dimensions were constructed in [75, 18, 19], while [11] classified the full set of BPS solutions for the special case of the five-dimensional $\mathcal{N} = 2$ supergravity theory which is the truncation of the $\mathcal{N} = 8$ theory (i.e. the theory is invariant under 8 instead of 32 supersymmetries). The latter require specifying a four-dimensional base metric that is restricted to be hyperkähler [76]. A particularly appealing class of hyperkähler manifolds are Gibbons-Hawking or multi-Taub-NUT geometries which are asymptotically $\mathbb{R}^3 \times \text{S}^1$ and for which we have explicit metrics. Moreover, it has been shown that the five dimensional solutions constructed using a Gibbons-Hawking base manifold [10] correspond to the four dimensional ones via the 4d/5d connection [16, 15, 77]

making them an interesting class of solutions to study [13].

The five dimensional solutions, although relatively complicated, are determined entirely in terms of $2b_2 + 2$ harmonic functions where b_2 is the second Betti number of the compactification Calabi-Yau, X ,

$$\begin{aligned} H^0 &= \sum_a \frac{p_a^0}{|\mathbf{x} - \mathbf{x}_a|} + h^0, & H^A &= \sum_a \frac{p_a^A}{|\mathbf{x} - \mathbf{x}_a|} + h^A, \\ H_A &= \sum_a \frac{q_A^a}{|\mathbf{x} - \mathbf{x}_a|} + h_A, & H_0 &= \sum_a \frac{q_0^a}{|\mathbf{x} - \mathbf{x}_a|} + h_0. \end{aligned} \quad (5.1)$$

Here the coordinate vector \mathbf{x}_a gives the position in the spatial \mathbb{R}^3 of the a 'th center with charge $\Gamma_a = (p_a^0, p_a^A, q_A^a, q_0^a)$ (note here A runs from $1, \dots, b_2$). The IIA interpretation of these charges is (D6,D4,D2,D0) wrapping cycles of X while in M-theory the charge vector is (KK,M5,M2,P). Note that the harmonics have $2b_2 + 2$ constants $h = (h^0, h^A, h_A, h_0)$ that together determine the asymptotic behaviour of the harmonics and hence the solutions. We will also have frequent occasion to use the notation $\Gamma = (p^0, p^A, q_A, q_0)$ to refer to the total charge $\Gamma = \sum_a \Gamma_a$.

The position vectors have to satisfy the integrability constraints

$$\sum_b \frac{\langle \Gamma_a, \Gamma_b \rangle}{|\mathbf{x}_a - \mathbf{x}_b|} = \langle h, \Gamma_a \rangle, \quad (5.2)$$

where we define the symplectic intersection product

$$\langle \Gamma_1, \Gamma_2 \rangle := -p_1^0 q_0^2 + p_1^A q_A^2 - q_A^1 p_2^A + q_0^1 p_2^0. \quad (5.3)$$

By summing (5.2) over a we find that the constants h have to obey $\langle h, \Gamma \rangle = 0$. Note that even once the charges of each center have been fixed there is a large space of solutions that may even have several disconnected components. In particular, the constraint (5.2) implies that the positions of the centers are generally restricted, defining a complicated moduli space of (generically) bound solutions. It will turn out that in order to yield bound states in the quantum theory a symplectic form, given in Sec. 5.6, must be non-degenerate on these moduli spaces of solutions.

The metric, gauge field and Kähler scalars of the solution are now given in terms of the harmonics by

$$\begin{aligned} ds_{5d}^2 &= 2^{-2/3} Q^{-2} [-(H^0)^2 (dt + \omega)^2 - 2L(dt + \omega)(d\psi + \omega_0) + \Sigma^2 (d\psi + \omega_0)^2] \\ &\quad + 2^{-2/3} Q dx^i dx^i, \\ A_{5d}^A &= \frac{-H^0}{Q^{3/2}} (dt + \omega) + \frac{1}{H^0} \left(H^A - \frac{Ly^A}{Q^{3/2}} \right) (d\psi + \omega_0) + \mathcal{A}_d^A, \\ Y^A &= \frac{2^{1/3} y^A}{\sqrt{Q}}, \end{aligned} \quad (5.4)$$

where $\mathbf{x}^i \in \mathbb{R}^3$ and ψ is an angular coordinate with period 4π , and the functions appearing satisfy the relations

$$\begin{aligned}
d\omega_0 &= \star dH^0, \\
d\mathcal{A}_d^A &= \star dH^A, \\
\star d\omega &= \langle dH, H \rangle \\
\Sigma &= \sqrt{\frac{Q^3 - L^2}{(H^0)^2}}, \\
L &= H_0(H^0)^2 + \frac{1}{3}D_{ABC}H^A H^B H^C - H^A H_A H^0, \\
Q &= \left(\frac{1}{3}D_{ABC}y^A y^B y^C\right)^{2/3}, \\
D_{ABC}y^A y^B &= -2H_C H^0 + D_{ABC}H^A H^B.
\end{aligned} \tag{5.5}$$

Here the Hodge star is with respect to the flat \mathbb{R}^3 spanned by the coordinates \mathbf{x}^i and D_{ABC} are the triple intersection numbers of the chosen basis of $H^2(X)$. Note that the only equation in (5.5) for which there is no general solution in closed form is the last one. In some cases, e.g. when $b_2 = 1$, it is even possible to obtain a solution, in closed form, to this equation.

The function Σ appearing in the metric in (5.4) is known as the entropy function. When evaluated at \vec{x}_a it is proportional to the entropy of a black hole carrying the charge of the center lying at \vec{x}_a . This follows from the Bekenstein-Hawking relation and the fact that Σ would determine the area of the horizon of a possible black hole at \vec{x}_a . If this area is zero then the center at \vec{x}_a does not have any macroscopic entropy and, if the associated geometry does not suffer from large curvature in this region, then there is no reason to believe stringy corrections will change this. It was shown in [12, 13] that in classical $\mathcal{N} = 8$ supergravity, zero entropy smooth centers carry charges that are half-BPS. The half-BPS nature of the charge vector follows from smoothness. One way to see this is, taking r to be the distance from a half-BPS center, the associated harmonic functions would lead to a scaling of the form $r^{-1/2}$ of the entropy function Σ (as opposed to the r^{-2} behavior near a black hole). Such half-BPS charge vectors, which we think of as “zero-entropy bits”, are identified by vanishing of the entropy function, as well as its first and second derivatives [78]. There will be some appropriate generalization of this condition to the general Calabi-Yau compactification, which we could analyze by studying the falloff of the metric near a center in the general Calabi-Yau. While in classical $\mathcal{N} = 8$ supergravity the “zero-entropy bits” have to be 1/2-BPS, it is conceivable that less supersymmetric centers carrying zero entropy could also serve as “atomic constituents” of black holes, with higher derivative corrections and stringy degrees of freedom making them smooth.

A center with a half-BPS charge $\Gamma = (1, p/2, p^2/8, p^3/48)$ corresponds to a single D6 brane wrapping the Calabi-Yau with all lower-dimensional charges induced by abelian flux. A configuration with a single such center can be spectral flowed (see e.g. [79, 26]) to a single D6 brane

with no flux and hence no additional degrees of freedom in the Calabi-Yau; thus “integrating out” the Calabi-Yau degrees of freedom does not generate an entropy and the associated five dimensional solution is smooth. As discussed in [13,80], “zero-entropy bits” can also be D4 and D2-branes with flux or D0 branes. Generically they can carry a “primitive” half-BPS charge vector, i.e. some number of D6-branes with fractional fluxes such that the induced D2, D4 and D0 brane charges are all quantized, but have no common factor. Note that if the D6-brane charge is $N > 1$, the solution is not strictly smooth – there is relatively mild orbifold singularity (R^4/Z_N) in the five-dimensional theory.

The properties of these solutions and the philosophy outlined in this paper motivate a conjecture similar to the one in [13]: Every supersymmetric 4D black hole of finite area, preserving 4 supercharges, can be split up into microstates made up of zero-entropy “atoms”. In the context of $\mathcal{N} = 8$ supergravity a somewhat more restrictive conjecture was stated in [13], adding further that the zero-entropy “atoms” would be 1/2-BPS (16 supercharges), and that the “atoms” were bound by mutual non-locality of their charges.

Multicenter configurations with every center constrained to be of the above form have been studied in [10,81,12,13] and numerous other works by the same authors. Note that the associated four dimensional solution can have singularities associated to Kaluza-Klein reduction on a non-trivial S^1 fibration. This highlights an important distinction: while the entropy, determined by replacing H^A in the definition of $\Sigma(H)$ with Γ^A , is a duality invariant notion, the smoothness of the resulting supergravity solution is not (see e.g. [82]). Thus we will assume that solutions with all centers that are “zero-entropy bits” (in the sense of vanishing “microscopic” entropy) are candidate “microstate geometries” (in the sense described in Section 2) even though they may have naked singularities in some duality frames.

From (5.4) and (5.5) it may seem that the solutions are singular if H^0 vanishes but this is not the case as various terms in Q and L cancel any possible divergences due to negative powers of H^0 (in fact, the BTZ black hole can, in the decoupling limit introduced in the next section, be mapped to such a solution with H^0 vanishing everywhere).

An additional complication is the fact that even solutions satisfying the constraint equation (5.2) may still suffer from various pathologies, most notably closed time-like curves (CTCs). For instance, the prefactor to the $d\psi^2$ term in the metric may become negative if Σ becomes imaginary²⁷. Unfortunately there is no simple criterion which can be used to determine if a given solution is pathology free. To fill in this gap [38] and [40] devised the *attractor flow conjecture*, a putative criterion for the existence of (well-behaved) solutions which we will describe in section 5.3.

An essential feature of these solutions is that they are stationary but not static. In particular they carry quantized intrinsic angular momentum associated with the crossed electric and magnetic fields of the dyonic centers [38]

²⁷As described in [16] this would also imply that the 4-dimensional metric associated with this 5-d solution (via the 4d/5d connection of [15]) becomes imaginary as Σ appears directly in the former.

$$\vec{J} = \frac{1}{2} \sum_{a < b} \frac{\langle \Gamma_a, \Gamma_b \rangle \vec{x}_{ab}}{r_{ab}}. \quad (5.6)$$

This will be important when quantizing the solution space as it is a necessary (but not sufficient) condition for the latter to be a proper phase space with a non-degenerate symplectic form. A solution space with vanishing angular momentum does not enjoy this property and must be completed to a phase space by the addition of velocities (see e.g. [83]).

5.2 Decoupling

Some specific five dimensional solutions obtained from an $\mathcal{N} = 2$ truncation of $\mathcal{N} = 8$ (i.e. compactification on T^6) have been studied using AdS/CFT [71, 73] by taking a decoupling limit of a dualized form of the solutions. This cannot be generalized to an arbitrary compact CY because the duality group of the latter is not known. To study the full class of solutions using AdS/CFT it is desirable to have a more general decoupling limit that can embed a larger class of these solutions in an AdS_3 throat. The limit we will describe can be taken in a proper $\mathcal{N} = 2$ theory and will yield an $\text{AdS}_3 \times \text{S}^2$ near horizon geometry [27]. This decoupling limit only works for solutions whose total charge does not contain any overall $D6/\text{KK}$ -monopole charge so the relevant CFT is essentially the MSW CFT²⁸. Although the latter is not under very good control it is nonetheless possible to determine, from the asymptotics of a given geometry, the CFT quantum numbers of the corresponding state. It is also possible to use general CFT properties to determine various quantities such as the number of states in a given charge sector.

In [27] the decoupling limit of the solutions described above is defined by taking $\ell_5 \rightarrow 0$ (ℓ_5 is the 5-d plank length) while keeping fixed the mass of $M2$ branes stretched between the various centers and wrapping the M -theory circle. In doing so we also fix the volume of the Calabi-Yau as measured in 5-d plank units and the length of the M -theory circle, R . Since the mass of such membranes is given by $m_{M2} \sim R \Delta \mathbf{x} / \ell_5^3$, the coordinate distances between the centers, $\Delta \mathbf{x}$, must be rescaled as ℓ_5^3 . Alternatively, we can see this limit as a rescaling of the 5-d metric under which the Einstein part of the action is invariant.

We now define new rescaled coordinates, x^i , and harmonic functions, H , as

$$x^i = \ell_5^{-3} \mathbf{x}^i \quad H = \ell_5^{3/2} \mathbf{H} \quad (5.7)$$

By restricting to the region of finite x^i we are essentially keeping the mass of transverse, open membranes fixed while rescaling the original coordinates, \mathbf{x}^i . One can see that, in these new variables, the structure of the solution (in terms of the harmonics) does not change in the

²⁸The “MSW” CFT is the theory that arises as the low-energy description of $M5$ -branes wrapping an ample divisor in the Calabi-Yau. It is an $N = (0, 4)$ superconformal field theory and it owes its name to the three authors of [5].

decoupling limit except for an overall scaling of the metric by a factor of ℓ_5^{-2} . The rescaled harmonics do take a new form, however,

$$\begin{aligned} H^0 &= \sum_a \frac{p_a^0}{|x - x_a|}, & H^A &= \sum_a \frac{p_a^A}{|x - x_a|}, \\ H_A &= \sum_a \frac{q_A^a}{|x - x_a|}, & H_0 &= \sum_a \frac{q_0^a}{|x - x_a|} - \frac{R^{3/2}}{4}. \end{aligned} \quad (5.8)$$

In particular note that all the constants have disappeared except the $D0$ -brane constant which now takes a fixed value (See section 5.3 for the expressions of h). Related to this is the fact that the asymptotic value of the moduli are forced to the attractor point, $Y^A \sim p^A$ (this corresponds to sending the 4-d Kähler moduli to $J^A = \infty p^A$).

Recall that the coordinate locations of the centers must satisfy the integrability constraint (5.2) and that this constraint depends on the value of the constants in the harmonic functions. As a consequence it is possible that, in taking the decoupling limit, some solutions cease to exist. For instance if we consider two $D4$ - $D2$ - $D0$ centers, Γ_a and Γ_b , then the constraint equation in the decoupling limit is

$$\frac{\langle \Gamma_a, \Gamma_b \rangle}{x_{ab}} = h_0 p^0 = 0 \quad (5.9)$$

Implying that x_{ab} , the inter-center distance, must be infinite (unless the charges are parallel, $\langle \Gamma_a, \Gamma_b \rangle = 0$) so one of the centers is forced out of the finite region of the rescaled coordinates, x^i , as we take the limit. We interpret this as implying that such centers cannot sit in the same decoupled AdS_3 throat.

Because only the $D0$ constant survives, centers without $D6$ charge are no longer bound together. A related fact, which will emerge presently, is that the solution space of such centers does not have a non-degenerate symplectic form (because, in the decoupling limit, intrinsic angular momentum is proportional to $D6$ dipole-moment) on it and hence cannot be quantized without the addition of additional degrees of freedom.

Even if a set of charges admits a solutions that satisfies the constraint equations in the decoupling limit the solution may develop other pathologies such as CTCs. To study these we may resort once more to the attractor flow conjecture of [38] which is argued, in [26], to extend to the decoupled solutions. We will not discuss this in any depth here except to note that the fact that the asymptotic moduli, Y^A , are forced to the attractor point for the total charge implies that only attractor flow trees which can be extended to trees starting from this value of the asymptotic moduli will survive the decoupling limit. Further discussion can be found in Section 5.3 and [26].

The decoupled solutions are asymptotically $\text{AdS}_3 \times \text{S}^2$ and their asymptotic form is given by

$$ds_{5D}^2 = -\frac{\rho^2}{4U^2}dv d\psi + \frac{U^{-4}}{4}\left[-R^2(d^0)^2 dv^2 + \mathcal{D}d\psi^2\right] + 4U^2\frac{d\rho^2}{\rho^2} + U^2\left(d\theta^2 + \sin^2\theta d\alpha^2\right) + \mathcal{O}\left(\frac{1}{\rho^2}\right), \quad (5.10)$$

$$A_{5D}^A = -p^A \cos\theta d\alpha - D^{AB}q_B d\psi + \mathcal{O}\left(\frac{1}{\rho^2}\right), \quad (5.11)$$

$$Y^A = \frac{p^A}{U} + \mathcal{O}\left(\frac{1}{r^2}\right). \quad (5.12)$$

where we have introduced some coordinate redefinitions. We first introduce standard spherical coordinates, (r, θ, ϕ) , on the base spatial \mathbb{R}^3 of the solution with the axis of the sphere (the z -axis) aligned with the $D6$ -dipole moment of the solution

$$\vec{d}^0 := \sum_a p_a^0 \vec{x}_a \quad (5.13)$$

For brevity we introduce the notation $d^0 = |\vec{d}^0|$ and $\vec{e} = \vec{r}/r$ so d^0 is the norm of the dipole moment and \vec{e} is a unit vector in the radial direction. We then make a further coordinate redefinition

$$v = t - \frac{R}{4}\psi, \quad \alpha = \phi + R d^0 \left(\frac{p^3}{3}\right)^{-1} v, \quad U^3 = \frac{p^3}{6} \quad \text{and} \quad \mathcal{D} = \frac{p^3}{3} \left(D^{AB}q_A q_B - 2q_0\right), \quad (5.14)$$

and define a new radial coordinate ρ via

$$\frac{\rho^2}{4U^2} = -\frac{U^{-4}}{2}R \left(\frac{e \cdot d^A D_{ABC} p^B p^C}{3} - \frac{p^A q_A d^0 \cos\theta}{3} \right) + \frac{R}{U}r, \quad (5.15)$$

Here \vec{d}^A is the $D4$ -dipole moment, defined analogously to \vec{d}^0 . To make a connection to the dual CFT the solution needs to be reduced along the S^2 to give a theory defined purely on AdS_3 with KK modes in representations of $SU(2)$. This procedure is reviewed in some detail in [84] with particular attention to the subtleties involved in reducing the 5-d Chern-Simons terms in the supergravity action.

The resultant metric is asymptotically AdS_3 and from this, and the asymptotic form of the gauge field, we can determine the charges in dual field theory (see [27] for details). In particular we find that

$$L_0 = \frac{(p^A q_A)^2}{2p^3} - q_0 + \frac{p^3}{24}, \quad (5.16)$$

$$\tilde{L}_0 = \frac{(p^A q_A)^2}{2p^3} + \frac{p^3}{24}.$$

and that the $SU(2)$ R-symmetry charge associated with a solution is determined entirely in terms of its $D6$ dipole moment

$$J_0^3 = -\frac{R^2 d^0}{8} \quad (5.17)$$

While (5.16) gives the charges expected from general considerations of the MSW CFT associated with the total charge Γ , the $SU(2)$ charge, J_0^3 , depends on a dipole moment and, as such, is absent in the single center solution. In fact, it is nothing more than the intrinsic angular momentum, J , defined in (5.6) which, in the decoupling limit, can be shown to be proportional to the $D6$ dipole moment.

In the next section we will see that this charge plays a distinguished role in the quantization of the system as its presence is necessary in order to have a non-trivial symplectic form on the phase space.

5.3 Split Attractors and State Counting

In [38] a conjecture is proposed whereby pathology-free solutions are those with a corresponding *attractor flow tree* in the moduli space. This conjecture was first posed for multicentered four-dimensional solutions so we will introduce some four dimensional terminology here. The four dimensional moduli, $t^A(\vec{x}) = B^A(\vec{x}) + iJ^A(\vec{x})$, are the complexified Kähler moduli of the Calabi-Yau. The relation between these moduli and their five dimensional counterparts can be found in [15] [26]. To each charge vector, Γ_i , we can associate a complex number, the central charge, as

$$Z(\Gamma_i; t) := \langle \Gamma_i, \Omega(t) \rangle \quad \Omega(t) := -\frac{e^t}{\sqrt{\frac{4}{3}J^3}} \quad (5.18)$$

Note that, since t^A is a two-form, Ω is a sum of even degree forms. The phase of the central charge, $\alpha(\Gamma_i) := \arg[Z(\Gamma_i; t)]$, encodes the supersymmetry preserved by that charge at the given value of the moduli. The even form Ω is related, asymptotically, to the constants in the harmonics (5.1) (which define both the 4-d and 5-d solutions) as

$$h = -2 \operatorname{Im}(e^{-i\alpha(\Gamma)} \Omega)|_\infty \quad (5.19)$$

For more details of the 4-d solution the reader should consult [19].

An *attractor flow tree* is a graph in the Calabi-Yau moduli space beginning at the moduli at infinity, $t^A|_\infty$, and ending at the attractor points for each center. The edges correspond to single center flows towards the attractor point for the sum of charges further down the tree. Vertices can occur where single center flows (for a charge $\Gamma = \Gamma_1 + \Gamma_2$) cross walls of marginal stability where the central charges are all aligned ($|Z(\Gamma)| = |Z(\Gamma_1)| + |Z(\Gamma_2)|$). The actual flow of the moduli $t^A(\vec{x})$ for a multi-centered solution will then be a thickening of this graph (see [38], [40] for more details). According to the conjecture a given attractor flow tree will

correspond to a single connected set of solutions to the equations (5.2), all of which will be well-behaved. An example of such a flow is given in figure 2.

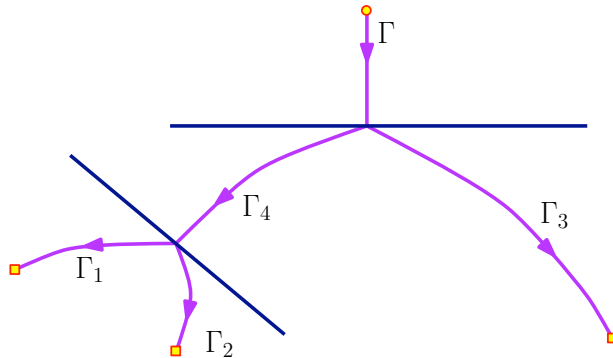


Figure 2: Sketch of a 3-center attractor flow tree from [40] [26]. The dark blue lines are lines of marginal stability, the purple lines are single center attractor flows. The tree starts at the yellow circle (the moduli at infinity) and flows towards the attractor points indicated by the yellow boxes. Note here that $\Gamma_4 = \Gamma_1 + \Gamma_2$ and $\Gamma = \Gamma_4 + \Gamma_3$. On the walls of marginal stability the moduli are such that $|Z(\Gamma; t)| = |Z(\Gamma_3; t)| + |Z(\Gamma_4; t)|$ (horizontal wall on top) and $|Z(\Gamma_4; t)| = |Z(\Gamma_1; t)| + |Z(\Gamma_2; t)|$ (diagonal wall on bottom left).

The intuition behind this proposal is based on studying the two center solution for charges Γ_1 and Γ_2 . The constraint equations (5.2) imply that when the moduli at infinity are moved near a wall of marginal stability (where Z_1 and Z_2 are parallel) the centers are forced infinitely far apart

$$r_{12} = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{\langle h, \Gamma_1 \rangle} = \frac{\langle \Gamma_1, \Gamma_2 \rangle |Z_1 + Z_2|}{2 \operatorname{Im}(\bar{Z}_2 Z_1)} \Big|_{\infty} \quad (5.20)$$

In this regime the actual flows in moduli space are well approximated by the split attractor trees since the centers are so far apart that the moduli will assume single-center behaviour in a large region of spacetime around each center. Thus in this regime the conjecture is well motivated. Varying the moduli at infinity continuously should not alter the BPS state count, which corresponds to the quantization of the two center moduli space, so unless the moduli cross a wall of marginal stability we expect solutions smoothly connected to these to also be well defined. Extending this logic to the general N center case requires an assumption that it is always possible to tune the moduli such that the N centers can be forced to decay into two clusters that effectively mimic the two center case. There is no general argument that this should be the case but one can run the logic in reverse, building certain large classes of solutions by bringing in charges pairwise from infinity and this can be understood in terms of

attractor flow trees. What is not clear is that all solutions can be constructed in this way. For more discussion the reader should consult [39].

Although this conjecture was initially proposed for asymptotically flat solutions, in [27] it was argued that the essential features of the attractor flow conjecture continue to hold in the decoupling limit.

For generic charges the attractor flow conjecture also provides a way to determine the entropy of a given solution space. The idea is that the entropy of a given total charge is the sum of the entropy of each possible attractor flow tree associated with it. Thus the partition function receives contributions from all possible trees associated with a given total charge and specific *moduli at infinity*. An immediate corollary of this is that, as emphasized in [40], the partition function depends on the asymptotic moduli. As the latter are varied certain attractor trees will cease to exist; specifically, a tree ceases to contribute when the moduli at infinity cross a wall of marginal stability (MS) for its first vertex, $\Gamma \rightarrow \Gamma_1 + \Gamma_2$, as is evident from (5.20).

For two center solutions one can determine the entropy most easily near marginal stability where the centers are infinitely far apart. In this regime locality suggests that the Hilbert state contains a product of three factors²⁹ [40]

$$\mathcal{H}(\Gamma_1 + \Gamma_2; t_{ms}) \supset \mathcal{H}_{\text{int}}(\Gamma_1, \Gamma_2; t_{ms}) \otimes \mathcal{H}(\Gamma_1; t_{ms}) \otimes \mathcal{H}(\Gamma_2; t_{ms}) \quad (5.21)$$

Since the centers move infinitely far apart as t_{ms} is approached we do not expect them to interact in general. There is, however, conserved angular momentum carried in the electromagnetic fields sourced by the centers and this also yields a non-trivial multiplet of quantum states. Thus the claim is that \mathcal{H}_{int} is the Hilbert space of a single spin J multiplet where $J = \frac{1}{2}(|\langle \Gamma_1, \Gamma_2 \rangle| - 1)$.³⁰ $\mathcal{H}(\Gamma_1)$ and $\mathcal{H}(\Gamma_2)$ are the Hilbert spaces associated with BPS brane excitations in the Calabi-Yau and their dimensions are given in terms of a suitable entropy formula for the charges Γ_1 and Γ_2 valid at t_{ms} .

Thus, if the moduli at infinity were to cross a wall of marginal stability for the two center system above the associated Hilbert space would cease to contribute to the entropy (or the index). A similar analysis can be applied to a more general multicentered configuration like that in figure 2 by working iteratively down the tree and treating subtrees as though they correspond to single centers with the combined total charge of all their nodes. The idea is, once more, that we can cluster charges into two clusters by tuning the moduli and then treat the clusters effectively like individual charges. We can then iterate these arguments within each cluster. This counting argument mimics the constructive argument for building the solutions by bringing in charges from infinity and is hence subject to the same caveats, discussed above.

²⁹Since attractor flow trees do not *have* to split at walls of marginal stability, there will in general be other contributions to $\mathcal{H}(\Gamma_1 + \Gamma_2; t_{ms})$ as well.

³⁰The unusual -1 in the definition of J comes from quantizing additional fermionic degrees of freedom [18] [27].

Altogether the above ideas allow us to determine the entropy associated with a particular attractor tree, which, by the split attractor flow conjecture corresponds to a single connected component of the solutions space. The entropy of a tree is the product of the angular momentum contribution from each vertex (i.e. $|\langle \Gamma_1, \Gamma_2 \rangle|$, the dimension of \mathcal{H}_{int}) times the entropy associated to each node. When we want to compare against the number of states derived from quantizing the classical phase space, however, the latter factor (from the nodes) will not be included as it is not visible in the supergravity solutions.

In the following sections we will show that it is also possible, in the two and three center cases, to quantize the solution space directly and to match the entropy so derived with the entropy calculated using the split attractor tree. This provides a non-trivial check of both calculations.

Before proceeding to count the number of states associated to an attractor flow tree we should mention an important subtlety in using the attractor flow conjecture to classify and validate solutions. Certain classes of charges will admit so called *scaling solutions* [80] [40] which are not amenable to study via attractor flows. These solutions are characterized by the fact that the constraint equations (5.2) have solutions that continue to exist at any value of the asymptotic moduli. We will discuss these solutions in greater detail in the next subsection but it is important to note here that the general arguments given in this section (such as counting of states via attractor flow trees) do not apply to scaling solutions.

5.4 Scaling Solutions

It was observed in [18] that for certain choices of charges it is possible to have points in the solution space where the coordinate distances between the centers goes to zero. Moreover, this occurs for any choice of moduli so it is, in fact, a property of the charges alone. Subsequently [80] noted that these “scaling solutions” develop deep throats as the coordinate distance between the centers decreases, but that the proper distance between the centers remains finite in the same limit. Scaling solutions constructed from three centers, each of them involving microscopic non-Abelian degrees of freedom, were emphasized in [13] as likely dominant components of the underlying state space. The non-Abelian degrees of freedom were supposed to arise in this picture from the open strings on the D-branes at each 1/2-BPS center, and [85, 86] appeared to confirm this picture by utilizing these degrees of freedom to allow D0-branes in the solution to polarize by a dielectric effect [87] into membranes. The authors of [85] argued that such microscopic configurations would dominate the entropy of the black hole with same total charge as the solution as a whole. The entropy of scaling solutions and their possible relation to single and multi-centered black holes was further explored in [40, 26].

Such solutions occur as follows. We take the inter-center distances to be given by $r_{ab} = \lambda \Gamma_{ab} + \mathcal{O}(\lambda^2)$ (fixing the order of the ab indices by requiring the leading term to always be positive). As $\lambda \rightarrow 0$ we can always solve (5.2) by tweaking the λ^2 and higher terms. The

leading behaviour will be $r_{ab} \sim \lambda \Gamma_{ab}$ but clearly this is only possible if the Γ_{ab} satisfy the triangle inequality. Thus any three centers with intersection products Γ_{ab} satisfying the triangle inequalities define a scaling solution.

We will in general refer to such solutions as *scaling solutions* meaning, in particular, supergravity solutions corresponding to $\lambda \sim 0$. The space of supergravity solutions continuously connected (by varying the \vec{x}_p continuously) to such solutions will be referred to as *scaling solution spaces*. We will, however, occasionally lapse and use the term scaling solution to refer to the entire solution space connected to a scaling solution. We hope the reader will be able to determine, from the context, whether a specific supergravity solution or an entire solution space is intended.

These scaling solutions are interesting because (a) they exist for all values of the moduli; (b) the coordinate distances between the centers go to zero; and (c) an infinite throat forms as the scale factor in the metric blows up as λ^{-2} . Combining (b) and (c) we see that, although the centers naively collapse on top of each other, the actual metric distance between them remains finite in the $\lambda \rightarrow 0$ limit. In this limit an infinite throat develops looking much like the throat of a single center black hole with the same charge as the total charge of all the centers. Moreover, as this configuration exists at any value of the moduli, it looks a lot more like a single center black hole (when the latter exists) than generic non-scaling solutions. As a consequence of the moduli independence of these solutions it is not clear how to understand them in the context of attractor flows; the techniques developed in [27] provide an alternative method to quantize these solutions that applies even though the attractor tree does not.

Unlike the throat of a normal single center black hole the bottom of the scaling throat has non-trivial structure. If the charges, Γ_a , are zero entropy bits (e.g. D6's with flux) then the 5-dimensional uplifts of these solutions will yield smooth solutions in some duality frame and the throat will not end in a horizon but will be everywhere smooth, even at the bottom of the throat. Outside the throat, however, such solutions are essentially indistinguishable from single center black holes. Thus such solutions have been argued to be ideal candidate spacetime realizations of microstates corresponding to single center black holes [80, 13]. In [40] it was noticed that some of these configurations, when studied in the Higgs branch of the associated quiver gauge theory, enjoy an exponential growth in the number of states unlike their non-scaling cousins which have only polynomial growth in the charges.

5.5 Quantization

As anticipated in [22], since the dual (0,4) CFT of $\mathcal{N} = 2$ black holes (lifted to 5-d) is less well understood than its (4,4) cousin, in this case one can hope to make progress by quantizing the phase space of the supergravity solutions directly. The quantization we will perform will be quite general in that it will cover the original 4-d multi-center black hole configurations [19], their 5-d uplift discussed in section 5.1, and the decoupled version of the latter (which can be

related to the (0,4) CFT). For $\mathcal{N} = 2$ black holes coming from Calabi-Yau compactifications it is likely that a large portion of the entropy will come from stringy degrees of freedom in the Calabi-Yau so it is likely to not be possible to account for a finite fraction of the entropy using supergravity alone. Whether this is the case or not is an important question though the answer may be difficult to determine as the solutions are rather complicated and even the classical phase space is quite non-trivial.

A picture where black hole microstates are realized in spacetime as extended bound states would associate to a given total charge, Γ , all possible decompositions of Γ into N charges, Γ_a , positioned at different centers, such that the solutions are smooth. The associated phase space for each decomposition would then be the (possibly disconnected) solution space for the charges. Quantization of this phase space should, in principle, yield certain microstates of the black hole and the set of (supergravity) microstates³¹ should come from summing over all possible decompositions. The notion of smoothness is not necessarily (duality frame) invariant so a more precise criterion might be that the constituent charges, Γ_a , should have no entropy associated with them (as discussed above).

For a given decomposition into N centers the phase space will be the $2N - 2$ dimensional submanifold of \mathbb{R}^{3N-3} given by solving the constraint equations (5.2) for the positions \vec{x}_a . Note that in arriving at this counting we have subtracted the 3 center of mass degrees of freedom; while these are present they generally decouple. As mentioned, this manifold may be disconnected and may possess a rather complicated topology. Moreover, for a given total charge, Γ , there will be many possible decompositions into different numbers of centers implying that the total phase space will be a disconnected sum of many manifolds of different dimension. Determining the symplectic structure of even the lowest dimensional solution spaces is already challenging [27].

5.6 Symplectic Form

In order to quantize the phase spaces described in section 5.5 we will need to determine the symplectic structure on these spaces. This can, in principle, be derived from the supergravity action as was done, for instance, in [21]. In this case, however, it is far more tractable to take a different approach [27]. As discussed in [18], the four dimensional multi-centered solutions can also be analyzed in the probe approximation by studying the quiver quantum mechanics of D-branes in a multicentered supergravity background. Moreover, a non-renormalization theorem [18] implies that the terms in the quiver quantum mechanics Lagrangian linear in the velocities do not receive corrections, either perturbatively or non-perturbatively. We can use this fact to calculate the symplectic form in the probe regime and extend it to the fully back-

³¹Actually, the full set of supergravity microstates probably requires considering more general solutions than those with Gibbons-Hawking base but one can at least hope that the latter set contributes a finite fraction of the entropy.

reacted solution; this is because, for time-independent solutions, the symplectic form depends only on the terms in the action linear in the velocity.

For this approach to be consistent it is necessary that the BPS solution space, which we interpret as a phase space, of the four and five dimensional supergravity theories, as well as the probe theory, all match. This follows from the fact that they are all governed by the same equation, (5.2) [18]. For instance, one can see that a probe brane of charge Γ_a in the background generated by a charge Γ_b is forced off to infinity as a wall of marginal stability is approached [18] analogous to what was described below equation (5.9) for the corresponding supergravity solutions.

In [27] the symplectic form on the solution space is determined. We will not review the derivation in detail but simply note that it arises from the term coupling the probe brane to the background gauge field, $\dot{x}^i A_i$, giving

$$\tilde{\Omega} = \frac{1}{2} \sum_p \delta x_p^i \wedge \langle \Gamma_p, \delta \mathcal{A}_d^i(x_p) \rangle. \quad (5.22)$$

where \mathcal{A}_d is the “spatial” part of the gauge field defined in (5.4) (this descends naturally to the spatial part of the 4-d gauge field). Using the definition of \mathcal{A}_d we can further manipulate this expression [27] and put it in the form

$$\tilde{\Omega} = \frac{1}{4} \sum_{p \neq q} \langle \Gamma_p, \Gamma_q \rangle \frac{\epsilon_{ijk} (\delta(x_p - x_q)^i \wedge \delta(x_p - x_q)^j) (x_p - x_q)^k}{|\mathbf{x}_p - \mathbf{x}_q|^3}. \quad (5.23)$$

This is a two form on the $(2N - 2)$ -dimensional solution space which is a submanifold of \mathbb{R}^{3N-3} defined by (5.2). Moreover, one can show that, on this submanifold, this form is closed and, in the cases we will investigate below, non-degenerate. Thus it endows the solution space with the structure of a phase space. Note that, as anticipated, the center of mass degrees of freedom do not appear in the symplectic form above and hence decouple in the quantization of the system. They will, in principle, yield an overall pre-factor in the partition function which we will not take into account.

Although the constraint equations (5.2) are invariant under global $\text{SO}(3)$ rotations these are nonetheless (generically) degrees of freedom of the system and this is reflected in the symplectic form. If we contract (5.23) with the vector field that generates rotations around the 3-vector n^i (i.e. we take $\delta x_{pq}^i = \epsilon^{ijk} n^j x_{pq}^k$) then the symplectic form reduces to

$$\tilde{\Omega} \rightarrow n^i \delta J^i \quad (5.24)$$

where J^i are the components of the angular momentum vector defined in (5.6).

This is nothing more than the statement that the components J^i are the conjugate momenta associated to global $\text{SO}(3)$ rotations. In general the symplectic form on any of our phase spaces³²

³²This does not hold for solution spaces with unbroken rotational symmetries, such as solution spaces containing only collinear centers or only a single center. In these cases some $\text{SO}(3)$ rotations act trivially, do not correspond to genuine degrees of freedom and do not appear in the symplectic form.

will have terms like the above coming from the global $SO(3)$ rotations, in addition to terms depending on other degrees of freedom. As advertised (5.24) implies that solution spaces with any $J^i = 0$ will have a degenerate symplectic form and will therefore not constitute a proper phase space³³.

5.7 Quantizing the Two-center Phase Space

The inter-center distance of a two center configuration is fixed in terms of the charges and the moduli at infinity but the axis of the centers can still be rotated so, neglecting the center of mass degree of freedom, we are left with a solution space that is topologically a two-sphere with diameter

$$x_{12} = \frac{\langle h, \Gamma_1 \rangle}{\langle \Gamma_1, \Gamma_2 \rangle}. \quad (5.25)$$

The symplectic form (5.23) is proportional to the standard volume form on the two-sphere and is entirely of the form (5.24) (note here that, as mentioned in footnote 32, collinearity of the solution implies that one $U(1) \subset SO(3)$ decouples). In terms of standard spherical coordinates it is given by

$$\tilde{\Omega} = \frac{1}{2} \langle \Gamma_1, \Gamma_2 \rangle \sin \theta d\theta \wedge d\phi = |J| \sin \theta d\theta \wedge d\phi. \quad (5.26)$$

We can now quantize the moduli space using the standard rules of geometric quantization. We introduce a complex variable z by

$$z^2 = \frac{1 + \cos \theta}{1 - \cos \theta} e^{2i\phi} \quad (5.27)$$

and find that the Kähler potential corresponding to $\tilde{\Omega}$ is given by

$$K = -2|J| \log(\sin \theta) = -|J| \log \left(\frac{z\bar{z}}{(1 + z\bar{z})^2} \right). \quad (5.28)$$

The holomorphic coordinate z represents a section of the line-bundle L whose first Chern class equals $\tilde{\Omega}/(2\pi)$. The Hilbert space consists of global holomorphic sections of this line bundle and a basis of these is given by $\psi_m(z) = z^m$. However, not all of these functions are globally well-behaved. By examining the norm of ψ_m given by

$$|\psi_m|^2 \sim \int d\text{vole}^{-K} |\psi_m(z)|^2 \sim \int d\cos \theta d\phi (1 + \cos \theta)^{|J|+m} (1 - \cos \theta)^{|J|-m} \quad (5.29)$$

one finds that ψ_m only has a finite norm if $m \geq -|J|$ and $m \leq |J|$. The total number of states equals $2|J| + 1$. This is in agreement with the wall-crossing formula up to a shift by 1. It can be shown that the inclusion of fermionic degrees of freedom will get rid of this unwanted shift [27].

³³As mentioned in footnote 32 this does not hold in the two center case where some $SO(3)$ directions decouple. There are also potential subtleties with solution spaces where $J = 0$ at a single point but we neglect these for now.

The integrand in (5.29) is a useful quantity as it is also the phase space density associated to the state ψ_m . According to the logic in [23, 25, 28] (reviewed in Section 2) the right bulk description of one of the microstates ψ_m should be given by smearing the gravitational solution against the appropriate phase space density, which here is naturally given by the integrand in (5.29). Since there are only $2|J| + 1$ microstates, we cannot localize the angular momentum arbitrarily sharply on the S^2 ; rather, it will be spread out over an area of approximately $\pi/|J|$ on the unit two-sphere. It is therefore only in the limit of large angular momentum that we can trust the description of the two-centered solution (with two centers at fixed positions) in supergravity.

5.8 Quantizing the Three-center Phase Space

The phase space of the three center case is four dimensional. Placing one center at the origin (fixing the translational degrees of freedom) leaves six coordinate degrees of freedom but these are constrained by two equations. This leaves four degrees of freedom, of which three correspond to rotations in $SO(3)$ and one of which is related to the separation of the centers.

This space is most easily visualized in the decoupling limit³⁴ for the case when one of the centers has no $D6$ brane charge. In that case the solution has an angular momentum vector J^i directed between the two centers with $D6$ charge and the orientation of the direction of this vector defines an S^2 in the phase space. The third center is free to rotate around the axis defined by this vector providing an additional $U(1)$, which we will coordinatize by an angle σ , fibred non-trivially over the S^2 . Finally the angular momentum vector has a magnitude which may be bounded from both below and above and this provides the final coordinate in the phase space. This construction is perhaps not the most obvious one from a coordinate space perspective but in these coordinates the symplectic form takes a simple and convenient form. This can also be used when all the centers have $D6$ charge but then the relation between these coordinates and the locations of the centers is less straightforward.

The symplectic form in these coordinates is (see [27] for a derivation):

$$\tilde{\Omega} = j \sin \theta d\theta \wedge d\phi - dj \wedge D\sigma \quad (5.30)$$

with $D\sigma = d\sigma - A$, $j = |\vec{J}|$, and $dA = \sin \theta d\theta \wedge d\phi$, so that A is a standard monopole one-form on S^2 . The gauge field A implements the non-trivial fibration of σ over the S^2 . A convenient choice for A is $A = -\cos \theta d\phi$ so that finally the symplectic form can be written as a manifestly closed two-form

$$\tilde{\Omega} = -d(j \cos \theta) \wedge d\phi - dj \wedge d\sigma. \quad (5.31)$$

³⁴We will work in the decoupling limit simply because in this limit \vec{J} is directly proportional to the $D6$ dipole moment so it is easier to visualize it. The general characteristics of the solution space described hold for all incarnations of the solutions we have considered – the 4-d solutions, their 5-d uplift and the decoupling limit of the latter. We will use the language of 4-d charges simply for brevity.

Let us consider the shape of the solution space spanned by coordinates $[\theta, \phi, j, \sigma]$. θ and ϕ are standard spherical coordinates with the latter defining a $U(1)$ that degenerates at $\theta = 0, \pi$. As has already been mentioned the angular momentum usually spans some range $j \in [j_-, j_+]$ though there are cases where, for fixed charges, the angular momentum spans two separate ranges resulting in a solution space with two disconnected components (which must be quantized separately). On the boundaries of these regions, at $j = j_-$ or $j = j_+$, the centers are collinear so the coordinate σ degenerates. Thus the phase space is a symplectic manifold with two $U(1)$ actions (corresponding to rotations in σ and ϕ) which degenerate at special points. In fact the manifold is a toric Kähler manifold and can thus be quantized using the technology of [88] [89].

This is done in detail in [27] and here we will report only the results. Using the technology of geometric quantization we can determine the basis of states spanning the Hilbert space (defined as the space of normalizable holomorphic sections of an appropriate line bundle over the phase space). It turns out that the number of such states is given by

$$\mathcal{N} = (j_+ - j_- + 1)(j_+ + j_- + 1). \quad (5.32)$$

In fact this is not quite correct as we have neglected fermionic degrees of freedom in defining our phase space. It is possible to include these degrees of freedom and quantize the resulting system [27] and this slightly changes the set of states. The final result becomes

$$\mathcal{N} = (j_+ - j_-)(j_+ + j_-) \quad (5.33)$$

This is the number we now wish to compare to the entropy determined by the wall crossing formula.

Let us consider the attractor flow tree depicted in figure 2. For the given charges, Γ_1 , Γ_2 , and Γ_3 , there are, in fact, many different possible trees but, in terms of determining the relevant number of states, the only thing that matters is the branching order. In figure 2 the first branching is into charges Γ_3 and $\Gamma_4 = \Gamma_1 + \Gamma_2$ so the degeneracy associated with this split is $|\langle \Gamma_4, \Gamma_3 \rangle|$ and the degeneracy of the second split is $|\langle \Gamma_1, \Gamma_2 \rangle|$ giving a total number of states

$$\mathcal{N}_{\text{tree}} = |\Gamma_{12}| |(\Gamma_{13} + \Gamma_{23})| \quad (5.34)$$

where we have adopted an abbreviated notation, $\Gamma_{ij} = \langle \Gamma_i, \Gamma_j \rangle$.

To compare this with the number of states arising from geometric quantization of the solution space, (5.33), we need to determine j_+ and j_- . Recall that the latter correspond to two different collinear arrangements of the centers and, in a connected solution space, there can be only two such configurations [27]. To relate this to a given attractor flow tree we will *assume part of the attractor flow conjecture*, namely that we can tune the moduli to force the centers into two clusters as dictated by the tree. For the configuration in figure 2, for instance, this implies we can move the moduli at infinity close to the first wall of marginal stability (the horizon dark blue line) which will force Γ_3 very far apart from Γ_1 and Γ_2 . In this regime it

is clear that the only collinear configurations are $\Gamma_1\text{-}\Gamma_2\text{-}\Gamma_3$ and $\Gamma_2\text{-}\Gamma_1\text{-}\Gamma_3$; it is not possible to have Γ_3 in between the other two charges. Since j_+ and j_- always correspond to collinear configurations they must, up to signs, each be one of

$$j_1 = \frac{1}{2}(\Gamma_{12} + \Gamma_{13} + \Gamma_{23}) \quad (5.35)$$

$$j_2 = \frac{1}{2}(-\Gamma_{12} + \Gamma_{13} + \Gamma_{23}) \quad (5.36)$$

j_+ will correspond to the larger of j_1 and j_2 and j_- to the smaller but, from the form of (5.33), we see that this will only effect \mathcal{N} by an overall sign (which we are not tracking carefully in any case). Thus

$$\mathcal{N} = \pm(j_1 - j_2)(j_1 + j_2) = \pm\Gamma_{12}(\Gamma_{13} + \Gamma_{23}) \quad (5.37)$$

which nicely matches (5.34).

Of course to obtain this matching we have had to assume the attractor flow conjecture itself (in part) so it does not serve as an entirely independent verification. Demonstrating this matching more carefully would help validate both methods of state counting [27].

5.9 More than Three Centers?

The symplectic form (5.23), when non-degenerate, defines a phase space structure on the solution space for an arbitrary number of centers. Analysing the solution space for a generic set of charges however is quite difficult as the constraint equations (5.2) imbue this space with a complicated geometric structure. In the two and three center case we were able to do this because the space had a toric structure. Fortunately there is a much larger class of charges that also enjoy this property; namely any configuration with two generic charges, Γ_1, Γ_2 , interacting with any number, N , of mutually BPS particles Γ_i .

From the constraint equation, (5.2),

$$\frac{\langle \Gamma_i, \Gamma_1 \rangle}{x_{1i}} + \frac{\langle \Gamma_i, \Gamma_2 \rangle}{x_{2i}} = \alpha \quad (5.38)$$

$$\frac{\langle \Gamma_1, \Gamma_2 \rangle}{x_{12}} + \sum_i \frac{\langle \Gamma_1, \Gamma_i \rangle}{x_{1i}} = \beta \quad (5.39)$$

it is clear that the $U(1)$'s around the \vec{x}_{12} axis do not appear in the constraint equations (i.e. the separation, x_{ij} , between mutually BPS centers decouples) so each new coordinate x_i comes with an additional $U(1)$ isometry.

Two interesting examples in this class were studied in [27]. The first case is obtained by setting all the $\Gamma_i = \Gamma_2$ (but leaving $\Gamma_1 \neq \Gamma_2$). This corresponds to having a single center Γ_1 surrounded by a gas of particles of charge Γ_2 which do not interact with each other and sit at a fixed distance $\langle \Gamma_2, \Gamma_1 \rangle / \alpha$ from the Γ_1 . From (5.2) the solution space can be determined to

be a product of S^2 's corresponding to the position of each of the Γ_i 's on a sphere centered at \vec{x}_1 . This “halo” configuration is interesting and has occurred before in the literature [18] [40] because it corresponds to a system with a non-primitive charge $(N+1)\Gamma_2$. This is, in fact, a two charge system rather than an $N+2$ charge system since $N+1$ centers have parallel charges. In this case [27] was able to use the technology of geometric quantization of toric manifolds to compute the degeneracy in this setting and match it to that computed using attractor trees [38] once more providing a nice consistency check between the two techniques.

A further generalization of this corresponds to setting $\Gamma_{1,2} = (\pm 1, p/2, \pm p^2/8, p^3/48)$ with the + and - corresponding to center 1 and 2 respectively and setting $\Gamma_i = (0, 0, 0, -q_i)$ (with $p, q_i > 0$ and $N = \sum_i q_i$). This system is of physical interest because the total charge corresponds to a D4-D0 black hole if we take $N > p^3/24$. Moreover, note that the charges we have selected satisfy $\Sigma(\Gamma_a) = 0$ so these geometries have no entropy associated with them and are candidate microstate geometries for the D4-D0 black hole.

Letting $I = p^3/6$ the regime $N < I/4$ is referred to as the polar regime where single-centered D4-D0 black holes do not exist. The regime $N > I/2$, on the other hand, correspond to the regime in which the total charges, $\Gamma_1, \Gamma_2, \Gamma_0 = \sum_i \Gamma_i$, are *scaling* in the sense defined in Section 5.4. Recall that in the scaling regime the charges can collapse to a single point in the solution space and form an infinitely deep throat that strongly resembles a black hole to an outside observer. Hence, for $N > I/2$, it is the scaling solutions that are the most likely candidate microstate geometries.

This system was studied in [85] [90] where it was argued that D0's should yield the leading contribution to the black hole entropy after they expand into elliptic D2's via the Myers effect. It is thus interesting to see how many states are captured by these smooth supergravity solutions. This computation was done in [27] and the entropy of these configurations was shown to have a leading behaviour of $N^{2/3}$ in the regime $N < I/2$. The entropy of a black hole in this regime, $I/4 < N < I/2$, is of the order $\sqrt{NI} \sim N$ so clearly these configurations cannot account for all the states.

It would be interesting to compute the entropy associated to these configurations in the regime $N > I/2$ and compare this with the entropy of a black hole in this regime.

5.10 Large Scale Quantum Effects: Scaling Solutions

Although it has long been understood how to account for the number of black hole microstates in string theory [2] this has generally been done in a dual field theory making it difficult to address some fundamental questions in black hole quantum mechanics such as information loss via Hawking radiation. For some microscopic black holes (such as those discussed in Section 4) the ability to dualize to an F1-P system has allowed for a more detailed analysis of the structure of the microstates. For these black holes it has been argued [69] that the average microstate is a highly quantum superposition of states with the corresponding spacetime a wildly fluctuating

“fuzzball”. The very interesting part of this claim is that these fluctuations extend over a region of spacetime circumscribed by the putative black hole horizon. The “metrics”³⁵ corresponding to the states in the superposition are all very different within the region which would be enclosed by a horizon in the naive black hole solution but they settle down very quickly to the same metric outside the horizon. Thus the remarkable claim of [69] is that the generic state in the black hole ensemble has quantum fluctuations over a large region of spacetime reaching all the way to the black hole horizon.

Unfortunately the black hole discussed in [69] is microscopic and has no horizon in supergravity (without higher derivative corrections); it would thus be very desirable to be able to demonstrate this type of behaviour in a system with a macroscopic black hole. In [27] an attempt was made to do exactly this. Scaling multicenter solutions can classically form arbitrarily deep throats that become infinitely deep in the strict $\lambda \rightarrow 0$ limit where the coordinate separation of the centers vanishes. We expect, however, that quantum effects will prohibit us from localizing the centers arbitrarily close together and will thus cap off the throat. What is remarkable about this is that the symplectic form, and hence the quantum exclusion principle, is not renormalized as we increase g_s (to interpolate between quiver quantum mechanics and gravity) so, even though gravitational effects increase the distance between the centers as the throat forms, the phase space volume stays very small. Thus gravitational back-reaction essentially blows up these quantum effects to a macroscopic scale. This is important not only because it is reminiscent of the large scale quantum fluctuations of the D1-D5 black hole but also because a smooth geometry with an infinite throat would be hard to understand in the context of AdS/CFT. Many solution spaces with a scaling point persist and continue to exhibit scaling behaviour even after we take a decoupling limit making all the solutions asymptotically $\text{AdS}_3 \times \text{S}^2$. This is problematic as general arguments suggest that an infinitely deep throat in a smooth geometry that is asymptotically AdS would imply a continuous spectrum in the CFT [91]. Thus it is comforting that the analysis of [27] reveals the infinite throat to be an artifact of the classical limit. Indeed, this is precisely the kind of phenomenon that was suggested in [70].

Before discussing this phenomena in more detail let us note some caveats. The states defined by quantizing the scaling solutions spaces are not necessarily generic black hole microstates (in fact, it is probable that such states require including additional stringy degrees of freedom in the phase space) so they may not reflect the behaviour of the actual black hole ensemble. Also, the symplectic form was computed in the gauge theory and extended to gravity via a supersymmetric non-renormalization theory; it would be more insightful to have a direct supergravity computation of the symplectic form. These caveats notwithstanding it is remarkable that these solutions exhibit quantum structure on a large scale even though they are smooth

³⁵Most of the relevant states are stringy states so the term metric is not really appropriate. A more precise statement would be expectation values of a profile of the string in the F1-P system. See e.g. [69, 4] for more details.

with everywhere small curvature. We will return to this point presently.

What is actually determined in [27] is the maximal depth of an effective throat generated by trying to localize a state as much as possible in the small λ region of the phase space (recall $\lambda \rightarrow 0$ is the scaling point where the centers coincide in coordinate space and an infinitely deep throat forms in the geometry). Specifically, a three center solution similar to the one described in the previous section with a pure fluxed D6- $\overline{\text{D6}}$ pair and a single D0 with charge $-N$ is considered in its lowest angular momentum eigenstate and the expectation value of the harmonic H_0 and the D6-D0 separation is computed. The latter is shown to be of order $\epsilon \sim N/I \geq 1/2$ implying that the centers cannot be localized arbitrarily close to each other so an infinite throat never forms. Rather a cap is expected to form at a scale set by the D0-D6 distance. A wave equation analysis for a scalar field in a simplified asymptotically AdS background with a capped throat of order ϵ reveals that the corresponding mass gap in the CFT goes as ϵ/c where $c = 6I$ is the central charge of the CFT. The expected result, from comparison with the D1-D5 system (see e.g. [4]), is a mass gap of order $1/c$ which is indeed what is found in this case as ϵ is bounded from below by $1/2$ (since $N \geq I/2$ for the solution to be scaling).

While the computation above is heuristic in many ways it yields two very important qualitative lessons. The first is that quantization of these solution spaces as phase spaces resolves several classical pathologies such as infinitely deep throats and also clarifies the issue of bound states (see [27] for a discussion of this). More importantly, however, it demonstrates that classical solutions may be invalid even though they do not suffer from large curvature scales or singularities. This is an important point so let us explore it further.

In this particular system the phase space structure of the supergravity theory can be related to that of quiver quantum mechanics by a non-renormalization theorem. In the latter the scaling solutions (at weak coupling) are analogous to electron-monopole bound states. Heisenberg uncertainty implies the minimum inter-center distance is of order $x_{ij} \sim \hbar$. Moreover because the solution space is a *phase space* rather than a configuration space the coordinates are conjugate to other coordinates rather than velocities so it is not possible to localize all coordinate directions with arbitrary precision by constructing delta-function states³⁶. Thus this quantity will have a large variance so $\delta x_{ij}/x_{ij} \sim 1$ for very small x_{ij} . At weak coupling this is nothing more than the standard uncertainty principle and is not particularly surprising.

What is surprising is that this behaviour persists even once gravity becomes strong and the centers backreact stretching the infinitesimal coordinate distance between them to a macroscopic metric distance. Moreover, in this regime the depth of the throat is extremely sensitive to the precise value of x_{ij} (see [91] for a numerical example) thus the large relative value of δx_{ij} translates into wildly varying depths for the associated throat. The associated expectation values for any component of the metric have an extremely large variance $\delta g/g$ and so cannot

³⁶Even if this were not the case delta function localized states have a large spread in momentum and would thus destroy any bound state.

possibly correspond to good semi-classical states. It is somewhat unusual to have classical configurations that cannot be well approximated by semi-classical states (i.e. those with low variance) but here this can be seen to follow from the very small phase space volume this class of classical solutions occupy [70].

5.11 Summary

The solutions described in this section constitute a sort of extended classical foam with complicated topological structure. The proposal of [3] translated into this context is that the classical finite area supersymmetric black holes of string theory are simply effective descriptions of complex, extended, horizon-free underlying bound states, of which the above classical solutions would be simple (probably non-generic) examples. This is in the same spirit as our discussion of BPS extremal black holes in AdS_5 and $\text{AdS}_3 \times S^3$ in previous sections of these notes. A piece of evidence in favor of this idea [18, 13]: one can show that in the $g_s \rightarrow 0$ limit the centers in these solutions flow together to form a single center bound state of D-branes with the charges of a black hole. Indeed, it is precisely these bound states that were originally counted by Strominger and Vafa to explain the entropy of supersymmetric black holes [2]. Turning this around, one might suggest a picture where *every* black hole microstates begins life at $g_s = 0$ as a ground state of an intersecting D-brane system, and that as the coupling is increased to attain a finite Newton constant the bound state increases in transverse size forming a sort of “stringy spacetime foam”. Further evidence come from the results of [27] that show that upon quantizing the subset of these states that can be realized in supergravity, the class of solutions that have a potential relation to black hole microstates will have quantum fluctuations at spatial distances far larger than the string scale. There are two missing pieces in the reasoning: (1) One would want to show that these considerations apply to the generic microstate, since it is not clear that the bound states described here are enough to account for the black hole entropy, (2) There is a need to reconcile the observations of an observer falling into a black hole horizon with the sort of picture described here. Nevertheless, perhaps an understanding involving extended underlying bound states will eventually shed light on an enduring puzzle: why should entropy be proportional to horizon area, rather than having some other functional dependence on the parameters of a black hole?

6 Conclusions

One obvious limitation of our discussion has been the restriction to extremal supersymmetric black holes. Although these are the most tractable, eventually we would like to deal with non-supersymmetric, realistic black holes that Hawking radiate. In interesting recent work a connection between non-supersymmetric black holes and interacting fluid dynamics was found [92, 93] which suggests that the type of dual descriptions in terms of free gases of particles that

we used in the supersymmetric cases may not be sufficient. This does not mean that such black holes cannot be studied using the approaches discussed in this review, but it does indicate that the nature of microstates as well as the extrapolation from weak to strong coupling is much more complicated. For some recent progress in obtaining Hawking radiation from a microstate point of view, see e.g. [94, 95]. See also [23] for a naive and qualitative description of the microstates of an AdS-Schwarzschild in the weakly coupled gauge theory.

A key feature of the idea that black holes are effective descriptions of underlying extended bound states is that these bound states should roughly have an extent that agrees with that of the black hole. In particular, they should grow as the string coupling is increased in the same way as the black hole horizon grows, a property emphasized and explicitly shown for the three-charge supertube in [96]. In [13] it was shown that a large class of multi-centered bound solutions with the same asymptotic charges as black holes shows precisely this sort of growth with the string coupling. Another useful hint comes from the fluctuations at large proper distances in solutions of the kind needed to give effective black hole behavior (Sec. 5 and [27]). It would be interesting to show whether the spacetime realization of the generic black hole microstate can grow in this way too, especially since such a growth would have important consequences for resolving the information paradox [33].

For large black holes, it is unlikely that the underlying microstates can be described in supergravity alone. To see this, recall that in AdS/CFT the states that are dual to single particle supergravity modes in the bulk are BPS states and their descendants. Therefore, one might expect that supergravity solutions only contain information about products of BPS operators and their descendants which are the duals of general multiparticle supergraviton states. Now in general, the phase in which AdS contains a thermal gas of supergravitons is separated from the phase in which the AdS space contains a black hole via a phase transition. This seems to indicate that supergravitons alone do not have enough degrees of freedom to account for the black hole entropy. For example, for $\text{AdS}_3 \times S^3$ one can show explicitly that this is the case [97]. Thus if it does turn out that all black holes are understood as the effective descriptions of extended bound states, the latter will likely involve many stringy degrees of freedom.

Along similar lines, it would be worthwhile exploring in more detail to what extent the multi-centered solutions we considered in section 5 can account for a finite fraction of the entropy of the corresponding macroscopic black hole. They are not the most general 5d supersymmetric solutions, as we took special hyperkähler manifolds as our base, and in addition we ignored Calabi-Yau excitations and stringy excitations. Even if they do not contribute a finite fraction of the entropy, one may wonder whether one may at least find some typical black hole microstates in this class. This question is very difficult to answer, because it is not clear what set of macroscopic observables we should precisely include in our definition of typicality. Finally, we should be careful to not view all smooth multi-centered solutions as associated to a single black hole. Many are more properly thought of as being associated to multicentered black holes.

One may expect that only the solutions which are described by a single centered attractor flow (this will include in particular many scaling solutions) are honest microstates of a single black hole [27]. Obviously, much more work is required in order to apply the philosophy outlined in this review in greater detail to macroscopic supersymmetric black holes. Another open problem is to provide a more detailed map between black objects in the bulk and ensembles in the boundary CFT. What distinguishes semiclassical ensembles from non-classical ones? What possible first laws of thermodynamics can these black object have? How many chemical potentials do we typically need to include in their dual description? Is there a natural way to describe multicentered bound states in the dual field theory? We have only begun to see some glimpses of answers to these questions in the examples we have described.

Among the many possible generalizations, we would like to mention the following results. For the $\text{AdS}_5 \times S^5$ case, significant progress has been made in identifying the space of 1/4 BPS geometries [98], and perhaps an extension of the state-geometry map to the 1/4 BPS case is now feasible. There was also recent progress in generalizing the notion of typical states from $N = 4$ SYM theory to more general four-dimensional SCFT's [30].

A key question of principle is what the infalling observer is supposed to see in the picture outlined in these notes. If the picture is correct, one might imagine that there is a tension with the usual scenario of an infalling observer unaffected at the horizon of a large black hole. However, it might be that there is some kind of “correspondence principle” such that classical infalling observers still effectively see the well-known black hole interior geometry until a short distance from the singularity. One might imagine this happening via a net cancellation of the many local interactions of a large object with a diffuse quantum background, leading to an effective description in the usual geometric terms.

7 Literature Survey

Since the literature in this subject is voluminous, we will not attempt a comprehensive review but will rather merely point the reader in the direction of many relevant works and attempt to give a sense of the status of various branches of the field. In so doing we will no doubt miss some important developments but hope that references to these are contained in the works cited below.

7.1 The D1-D5(-P) and Related Systems

i- The D1-D5 system

This configuration arises in type-IIB string compactified on $\mathcal{M} \times S^1$, where \mathcal{M} is a 4 dimensional Ricci-flat manifold. The supergravity solution describes N_5 D5-branes wrapping the full compact space and N_1 D1-branes wrapping the S^1 ; the most general metric

corresponding to this configuration was constructed in [9] (also see [99, 100, 101]). Originally there were neither internal nor fermionic excitations. In [8], the internal excitations were included, and [67] included the fermionic excitations. The near horizon geometry of this system is $\text{AdS}_3 \times \text{S}^3 \times \mathcal{M}$, so the dual theory is a CFT_2 . It was in the context of this system that the fuzzball proposal was first proposed [3].

A nice, but slightly out-dated, review of the D1-D5 system and the fuzzball proposal is contained in [4]. Significant progress has occurred since this review: the supergravity phase space has been directly quantized [47, 48] (see also [107]), and the idea that black holes are simply effective geometries has been studied in detail in the context of this system [29, 102, 103, 25, 104, 70, 68]. A more up-to-date review, with an emphasis on AdS/CFT tests of the proposal, can be found in [105].

Supporting evidence for the stretched horizon idea was given in [102, 70]. In [102] it was argued that quantum gravity effects become important at scales larger than the Planck/string scales. In [70] a sub-ensemble of the original thermal ensemble was considered. It was shown that, even in this case, the area of stretched horizon matches the microscopic entropy of the subsystem to leading order. This continues to hold even after the inclusion of 1-loop string corrections. In [29, 25, 104, 68] various quantities in the CFT and supergravity were compared.

There are also studies of microstates using a 1/4-BPS probe approach, see e.g [106, 43, 107, 108].

ii- Beyond the D1-D5 system

Further generalizations of the D1-D5 system followed. They can be collected under two general themes: adding more charges and breaking supersymmetry. In the first case the aim was to add some charges (momentum most of the time) to reduce the amount of preserved supersymmetry without breaking all the supersymmetry. An incomplete list of references in this direction includes [109, 14, 110, 111, 112, 113, 114, 115]. The D1-D5-P system is also covered in the review [105].

In [109] a first attempt to construct dual geometries for D1-D5-P microstates was undertaken; here the momentum was added as a small perturbation of the D1-D5 system. The first success was achieved in [14] which was followed by other works [110, 111, 115]. In [114] some arguments were given to support the claim that higher order correction to these 3-charge geometries would not generate a horizon or a singularity if they were not present at tree level.

A class of 5-dimensional smooth solutions, closely related to the D1-D5-P system, have also been focus of an extended research program [71, 11, 12, 10, 81, 80, 13, 116, 85, 86, 91, 26, 82, 117, 27]. These solutions are discussed in section 5 of this review. A recent review of some of this work is [118].

On the other hand, less work has been done on non-BPS solutions [119,120,121,122,94]. In [119] smooth non-supersymmetric geometries were constructed. They were then proven to be classically unstable in [120]. This classical instability was shown to give rise to Hawking radiation in [94]. More details appeared in [123]. Other properties of these solutions were studied in [121]. Another set of non-supersymmetric solutions in 4 dimensions appeared in [122]. Another interesting study was the tunneling of a collapse of a shell to fuzz-ball geometries [124] which is needed for a possible fuzz-ball like proposal for more realistic four dimensional black holes.

iii- Bound versus unbound systems

To avoid overcounting the states responsible for a black hole’s entropy one needs to know that the solution one is dealing with describes a bound system. For the D1-D5 system, this can be achieved by adding D3 charge. This system has been studied in [125] in the context of the fuzzball proposal.

Other works in this direction are [112,103,82]. In [112] known D1-D5 (D1-D5-P) geometries were rewritten in a *fiber* \times *base* form in order to gain some insight into their structure. In [103] another route was followed. Using the D1-D5 system as a prototype, this paper studied the dynamical behavior of such systems and put forth a conjecture to distinguish bound systems. In [26] three charge systems were studied in an $\text{AdS}_3 \times \text{S}^2$ decoupling limit and a criterion for bound configurations was proposed in terms of split attractor trees [38]. In [27] some of these solutions were quantized allowing a direct analysis of whether the relevant state is bound or not. In [82] another criterion was given using spectral flow in $\text{AdS}_3 \times \text{S}^3$.

7.2 Asymptotically $\text{AdS}_5 \times \text{S}^5$ and Related Solutions

In $\text{AdS}_5 \times \text{S}^5$ one can, in principle, consider 1/2, 1/4, 1/8 and 1/16 excitations of the full string theory. The only known example of a supersymmetric black hole in AdS_5 with finite area is the Gutowski-Reall black hole [126,51] and this solution is only 1/16 BPS. There are 1/2 BPS black solutions [75,53,52,54], known as superstars, but these have a no horizon in two-derivative supergravity. Nonetheless, significant effort has gone into studying the possible smooth, asymptotically AdS_5 geometries with varying amounts of supersymmetry. For the 1/2 BPS case the solutions were completely classified in [7].

i- The 1/2 BPS Case

Various generalizations of the LLM [7] geometries have been considered. The notion of typical states in an ensemble was explored in [23], while in [28] a “metric” operator was defined in the CFT and used to establish a criterion to determine which states will have a well-defined classical dual geometry. Another direction of research, closely related to

topics discussed in this review, involved quantization of the original LLM geometries using phase space techniques [21, 22].

Thermal properties and Wilson loops in the CFT dual to these geometries were studied in [127, 128] while in [129, 130, 131] $1/2$ BPS geometries corresponding to defects in the CFT were considered. In [132] the relation between phase space densities in the fermion formulation of the theory and generalized Young tableaux is studied. In [133] the on-shell action for the LLM geometries was derived.

In [134] where different methods were used to calculate the entropy of a “black hole” resulting from coarse graining over LLM geometries. There was a perfect agreement between entropies calculated by CFT and gravity coarse grainings .

There is, in fact, a large volume of literature on the $1/2$ BPS case and our short survey is by no means intended to be comprehensive. For more references the reader should consult the works cited above.

ii- Less SUSY

Backgrounds which preserve only $1/4$, $1/8$ and $1/16$ supersymmetry have also been considered and explicit supergravity solutions have been constructed. For instance smooth $1/4$ BPS solutions were constructed in [135, 136, 137, 138] and an LLM-like prescription to derive them from droplets on the plane is related to constraints on brane webs in [139, 98].

Probe solutions (giant gravitons) preserving $1/8$ of the supersymmetry were studied in [140, 141, 142, 143, 144] and the back-reaction of such probes was worked out in [145, 146]. The $1/8$ BPS sector of the dual CFT was explored in [147].

As mentioned above, the $1/16$ BPS sector is distinguished by having black holes with macroscopic horizons [51, 126, 148, 149, 150]. The $1/16$ sector of the CFT has also been studied; operators potentially related to the black hole have been identified [151] and the entropy has been (qualitatively) reproduced [152]. Giant gravitons preserving $1/16$ of the supersymmetry were found in [153].

Attempts to treat the full set of $1/2$, $1/4$, and $1/8$ solutions in a common framework can be found in [154, 155]. Other related work includes [156, 157, 158]. Once more, this list is only intended to serve as an introduction to the literature and no doubt has failed to include many important works.

iii- No Supersymmetry

Another direction of investigation involves breaking all supersymmetries. Some research in this direction includes [159, 160, 161].

The considerations described above for the D1-D5 system and the $1/2$ -BPS states of AdS_5 were applied to other backgrounds (an M-theory solution) in [162]. In [31], general considerations are applied to study the differences between correlators of an operator in a typical state

and a thermal state. In this paper it is also shown that for a system with an entropy S , the variance in finitely local correlation functions over the entire Hilbert space will be suppressed by e^{-S} . Because of this, regardless of the detailed origin of black hole entropy, if there is *any* statistical interpretation of the underlying degeneracy semiclassical observers will have difficulty telling apart the microstates.

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